The Infinite Resistor Chain

Background to "Morseman" Column, November-December 2007
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Figure 1 (a) shows the original network of the problem set in the column. This string of $1\,\Omega$ resistors is infinitely long. What is the resistance seen looking between the terminals A and B?
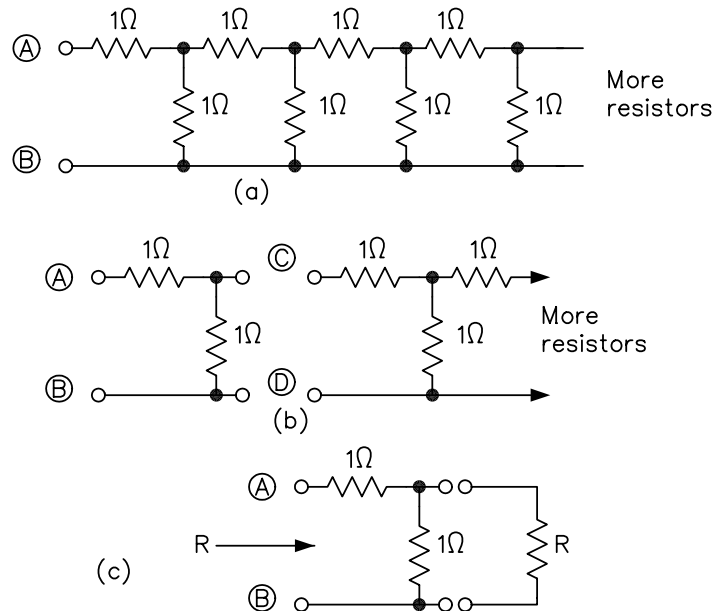

Figure 1: (a) Original network. (b) Network with first section isolated, for "standard" solution.

The "Standard" solution

Since there are infinitely many resistors, there will still be infinitely many if we detach the two resistors of the front section as shown in figure (b), and the resistance seen looking to the right between C and D will be the same as the resistance seen between A and B. Let this resistance be R.

Now replace the network to the right of C and D with its equivalent resistance, R, as shown in figure (c). The resistance seen between A and B will also be R. This is the resistance of $1\,\Omega$ in series with $1\,\Omega$ and R in parallel.

$$
\begin{aligned}
R &= 1 + \frac{1 \times R}{1 + R} = 1 + \frac{R}{1 + R} \\
&= \frac{1 + R + R}{1 + R}
\end{aligned}
$$

$$
\begin{aligned}
\text{cross-multiplying,} \quad R(1 + R) &= 1 + 2R \\
\text{rearranging,} \quad R^2 - R - 1 &= 0
\end{aligned}
$$

This is a quadratic equation, which can be solved by the standard method.[1] This gives

$$
R = \frac{1}{2} \pm \frac{\sqrt{1+4}}{2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}
$$

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[1]If $R^2 + AR + B = 0$, then $R = \left(A^2 - \sqrt{4C}\right)/2$

As uusual, there are *two* solutions. But $\sqrt{5}/2 = 1.118$, which is greater than a half. Hence we must choose the *positive* signed root to get a positive solution, since R, a resistance, must be a positive number. This gives $R = 1.6180340\ldots\,\Omega$, the solution required. The decimal doesn't terminate.

Second Solution: From Brian ZL1LE

Brian said:

"Hi Gary, I did the original solution on the back of an envelope!! I divided the circuit up into a series of Pi networks converting them into Stars (DRY - Delta to Star transformation using resistances) reducing the network down into another series of Pi networks, etc. Using Prof Bogle's strategy of using first principles and not trying to remember the mathematical series. hi hi!!"

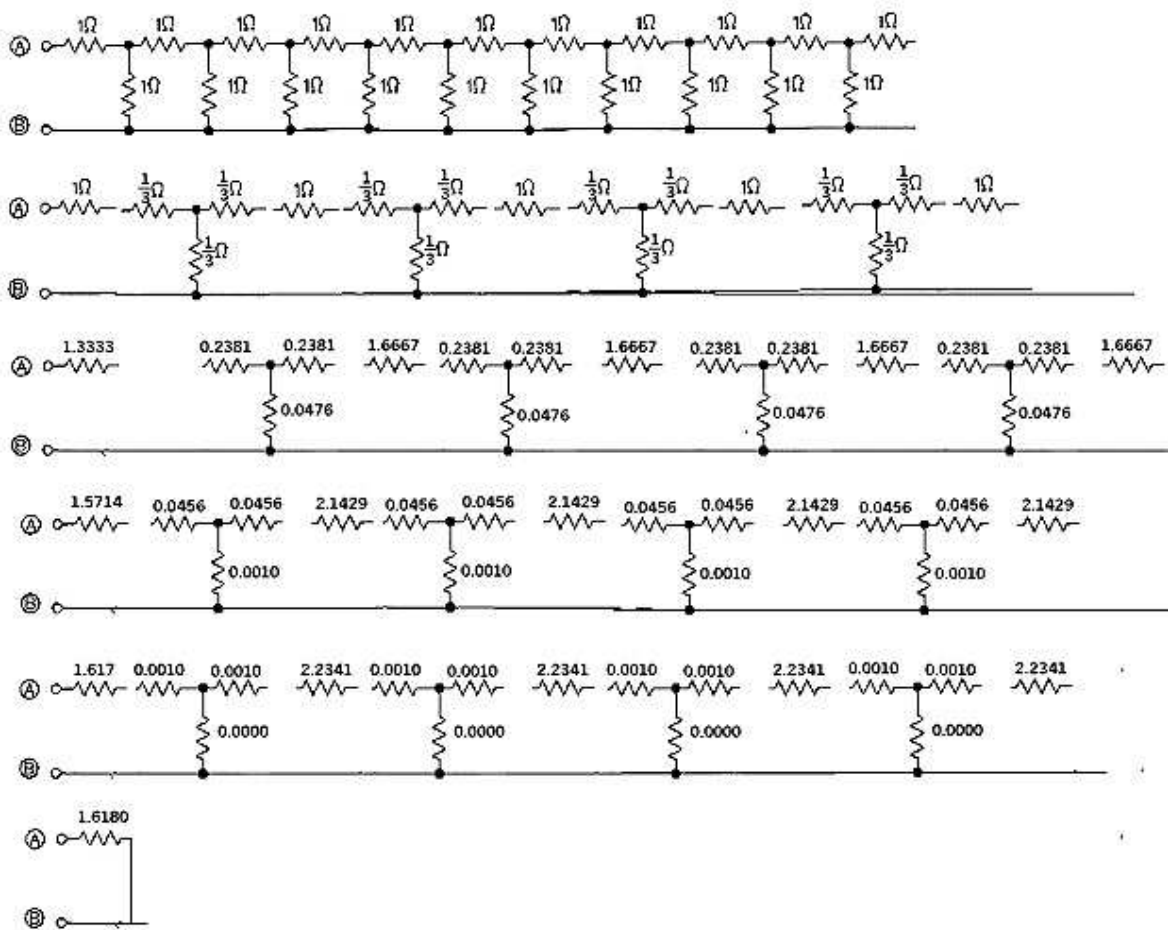See the diagram below, copied from Brian's email:

Figure 2: The progressive steps in Brian's solution.

The first line shows the original network. In the second line, the Pi network formed by the second, third and fourth resistors has been converted into a Tee. Let the conductances of these Pi network resistors (their reciprocal values) be G_1, G_2 and G_3. The standard Pi to Tee relations are

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$$R_1 \;=\; \frac{G_3}{G_1 + G_2 + G_3}, \qquad R_2 = \frac{G_2}{G_1 + G_2 + G_3}, \qquad R_3 = \frac{G_1}{G_1 + G_2 + G_3},$$

where $R_1, R_2, R_3 \;=\;$ the resistances in the corresponding Tee section.

Two things happen: The number of shunt resistors is now halved, and each shunt resistor has decreased in value. There are now more series resistors, whose values can be added.

In the third line, Pi to Tee conversions have been performed again, on the Pi's that resulted from the first operation. The shunt resistors get even smaller. Two more such steps are done. Now look at the front of the resulting network.

The shunt resistor, to four significant figures, has been reduced to $0.0000\,\Omega$, which means it's shorting out the rest of the network, which can be discarded! So the network impedance is just the sum of the input series resistances, or $1.6180\,\Omega + 0.0010\,\Omega = 1.680\,\Omega$.

Brian stopped here. If he had continued iterations, to more decimal places, he would have found $R = 1.6180340\ldots\,\Omega$, the exact solution required.

Third Solution: Stuart ZL2AAC

Now this is really interesting! Stuart said:

"I did it by considering the effect of a short-circuit across each, in turn, of the 'parallelled' resistors, ie those connected to the bottom line, and calculated the effective resistance of the shortened network at each step. Obviously, the more resistors considered, the more accurate the result. Happily, a pattern soon emerged, making further calculations simple." Here's a summary of Stuart's calculation:

$$
\begin{aligned}
\text{First resistor shorted: } R &= 1 \\
\text{second resistor shorted: } R &= 1 + \frac{1}{1+1} = \frac{5}{3} \\
\text{third resistor shorted: } R &= 1 + \frac{1}{1 + \dfrac{1}{1+1}} = \frac{5}{8} \\
\text{fourth resistor shorted: } R &= 1 + \frac{1}{1 + \dfrac{1}{1 + \dfrac{1}{1+1}}} = \frac{8}{13}
\end{aligned}
$$

Continuing (somewhat laboriously) in this manner, Stuart found that the series of progressively more accurate estimate of the input resistance was:

$$1, \quad \frac{5}{3}, \quad \frac{8}{5}, \quad \frac{13}{8}, \quad \frac{21}{13}, \quad \frac{34}{21}, \quad \frac{55}{34} \ldots \frac{233}{144}$$

Now the pattern emerges! If one estimate of the resistance is $\dfrac{a}{b}$, the next estimate will be $\dfrac{a+b}{a}$.

The last estimate in this series (tenth shunt resistor shorted) was $R = \dfrac{233}{144} = 1.61803\,\Omega$, which is the correct value to 6 decimal places.

The fascinating thing about this, is that the sequence of values that emerges is *exactly* that which defines the famous "Golden ratio", known mathematically as ϕ, and which is related to the Fibonacci

sequence! This was pointed out to me later by a colleague in the University of Auckland Physics Department, after examining Stuart's solution.

In fact, the iterated sequence of steadily increasing combinations of reciprocals which arise naturally in this solution is one of the standard sequences for calculating ϕ - although there are others which converge faster.

Here's an extract from the Wikipedia article on the Golden ratio:

"The mathematics of the golden ratio, ϕ, and of the Fibonacci sequence are intimately interconnected.

"Recall that the Fibonacci number sequence is:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ... The explicit expression for the Fibonacci sequence involves the golden ratio.

"The golden ratio is the limit of the ratios of successive terms of the Fibonacci sequence (or any Fibonacci-like sequence).

"Therefore, if a Fibonacci number is divided by its immediate predecessor in the sequence, the quotient approximates phi; e.g., 987/610 = approx 1.6180327868852. These approximations are alternately lower and higher than phi, and converge on ϕ as the Fibonacci numbers increase ..."

Hence Stuart's solution revealed a fascinating property of this infinite network of $1\,\Omega$ resistors, which was quite unknown to me or my colleagues. Even though I have set this problem many times to my network theory classes as an example requiring lateral thinking, the significance of the numerical value of the answer had never occurred to me!

Fourth Solution: Wilbert, ZL2BSJ

Wilbert said:

"I was unable to come up with a mathematical expression so I scripted the network in Octave (see below). After only a few iterations, the resistance between points A and B started to approach 1.618 ohms."

```
#!/usr/bin/octave -qf
n = 1
r = 2 do
      n++
      r = 1 + 1/(1+1/r)
until n >= 100
```

This converges to $r = 1.618033988749895$, accurate to 16 significant figures, after 20 iterations. This is a "brute force" solution, but it shows how an elegant piece of code can compute the value to arbitrary accuracy.

Fifth Solution

This is also a brute force method. You can numerically calculate the input impedance of *any* circuit by just simulating it in any variant of `Spice`. For example, here's the schematic of the circuit in `LTspice`.
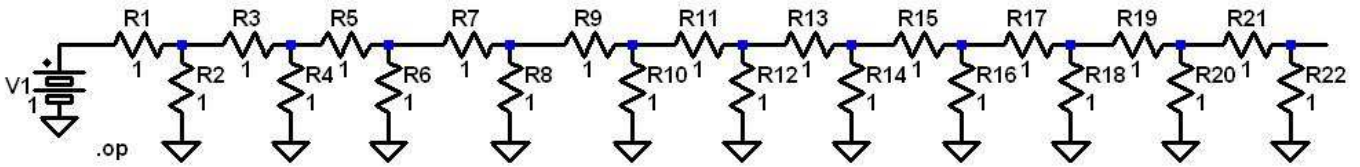

Figure 3: Circuit schematic drawn in `LTspice`.

The network is energized by a 1V battery, V_1. The input current is found as

$$I_{in} = 0.618034 \text{ Amp}$$
$$\text{whence } R_{in} = \frac{V_{in}}{I_{in}} = \frac{1}{0.618034}$$

$$R_{in} = 1.618034\,\Omega$$

Notice that the decimal figures are identical in both the current and its reciprocal! This is also a property associated with the Golden sequence.

Of course, with a `Spice` simulation, you can easily look at the voltages and currents in all parts of the circuit. We find that the voltage across the last resistor (R22) is only $34.9\,\mu$V, or the current through it is only $34.9\,\mu$A. So clearly adding additional resistors beyond this point, (11 sections) makes very little difference.

Nobody submitted this method of solution, but it's one that some members of my Network theory classes have used when they can't figure it out, but are desperate to get a solution!

Conclusion

The only method which gives an arbitrarily exact solution is the first, "classical" one, which I have always used. The others require an infinite series of iterations for progressively more accurate answers, but are very interesting nevertheless.

The prize, if there was one, would go to Stuart, whose solution revealed a hitherto unsuspected property of the answer. What does it mean? Read the Wikipedia article for more information. The "golden ratio" pops up all over the place.

Interesting aside: If you replace all of the $1\,\Omega$ shunt resistors with $2\,\Omega$ resistors, the input resistance *also* becomes *exactly* $2\,\Omega$.