Understanding the Goldman Equations for the Transient Analysis as They Apply to the Quarter Wave Matching Transformer

[Stanford Goldman’s “Transformation Calculus and Electrical Transients”]

By Robert Lay
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Introduction

In 1949, Stanford Goldman, Professor of Electrical Engineering at Syracuse University, wrote his book, “Transformation Calculus and Electrical Transients”, which was published by Prentice-Hall, Incorporated as part of their Electrical Engineering Series, edited by W. L. Everitt. I was given a copy of this book by my late friend and colleague, Sukru Cafer Durusel, who died in July 1965. My interest in the problem of explaining where the reflected power goes began in 1995 when my interest in RF transmission lines was re-awakened after returning to Amateur Radio after an absence of 25 years. I found that the equations provided in the ARRL Antenna Handbook and in the ITT Handbook for Radio Engineers were more than adequate to explain the impedance seen at any point on a transmission line, given its characteristic impedance and its terminating impedance. I also found that there was considerable disagreement in the community on certain aspects of the theory of standing waves. In particular, the condition in which a transmission line has an impedance discontinuity at both the source end and the load end. The standing wave pattern on such a line poses an enigma in that the reflected wave energy appears to simply stop and vanish at the source end of the line. There have been innumerable theories and explanations posed as where that power goes, and no theory appears to satisfy everyone. I have written two articles, one addressing the problem in the transient phase and the other addressing the problem in the steady-state. I found that Goldman’s work on transmission lines is the only published work that covers both aspects of the problem and provides a unified mathematical basis for each. The objective of these two articles is to try to bring the mathematical concepts outlined by Goldman to bear on specific, well known examples of the quarter wave matching transformer. The reader will find that no matter whether the transient or the steady-state model is pursued, the Goldman equations will provide a clearer picture of what really goes on in the transmission line. Each article begins with an explanation of the mathematical basis, which is then followed by a numerical example for a specific quarter wave matching transformer.

This paper addresses Chapter 10 of the subject book – in particular, the transient analysis of a terminated transmission line.

Figure 10-4 shows the simple transmission line circuit with its source generator and a complex load. The objective is to provide an analysis of the transient phase of the voltage and current as a function of time for any distance “x” along the transmission line of length “d” during startup transient conditions. Our discussion will apply directly to a quarter wave matching transformer, where $d = \lambda / 4$.

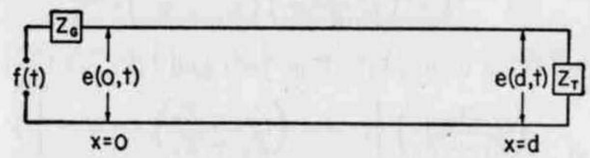

FIG. 10-4. A finite line with terminal impedances.

The fundamental equations which govern phenomena in a transmission line are

$$L\frac{\partial i}{\partial t} + Ri = -\frac{\partial e}{\partial x} \qquad (1)$$

$$C\frac{\partial e}{\partial t} + Ge = -\frac{\partial i}{\partial x} \qquad (2)$$

The two partial differential equations are an expression of Kirchoff's law for an infinitesimal length of line having series inductance of L Henries per meter, series resistance of R ohms per meter, shunt capacitance of C Farads per meter and conductance G mhos per meter. Equation (1) says that the Kirchoff voltage loop consists of the self induced voltage in volts per meter, the series resistance drop in volts per meter and the total voltage change per meter. Likewise Equation (2) says that the Kirchoff current summation consists of the capacitive charging current in amperes per meter, the conductance or leakage current in amperes per meter and the total shunt current per meter.

Note that the observation point, x, is the distance from the generator terminals in the direction of the load.

The objective is to find solutions to these simultaneous partial differential equations in the form

$$i = i(x,t)$$

$$e = e(x,t)$$

If we multiply through Equations (1) and (2) by e^{-st} and integrate from 0 to infinity we obtain the Laplace transformed equations corresponding to Equations (1) and (2)

$$[Ls + R]\bar{I} = -\frac{\partial \bar{E}}{\partial x} + Li(x,0) \qquad (9)$$

$$[Cs + G]\bar{E} = -\frac{\partial \bar{I}}{\partial x} + Ce(x,0) \qquad (10)$$

where $\bar{I} = \bar{I}(x,s)$ and $\bar{E} = \bar{E}(x,s)$ are the Laplace transforms of $i = i(x,t)$ and $e = e(x,t)$, respectively.

Equations (9) and (10) can be solved as a pair of simultaneous ordinary differential equations. Multiplying (9) by (Cs + G) and differentiating (10) with respect to x and adding the results, we obtain

$$\frac{\partial^2 \bar{I}}{\partial x^2} - [Ls + R][Cs + G]\bar{I} = C\frac{\partial e(x,0)}{\partial x} - L(Cs + G)i(x,0) \qquad (11)$$

$$\frac{\partial^2 \overline{E}}{\partial x^2} - [Ls+R][Cs+G]\overline{E} = L\frac{\partial i(x,0)}{\partial x} - C(Ls+R)e(x,0) \qquad (12)$$

There are several different problems that can now be solved. Our interest lies with the finite, terminated line of Figure 10-4.

An assumption that would make the equations much simpler would be for the line to be distortionless, which requires that $\frac{R}{L} = \frac{G}{C}$. However, we will go an additional step and also require the line to be lossless, which requires that R = G = 0.

With those simplifications and also assuming initially quiescent conditions (which cause the terms on the right of (16) and (17) to vanish), and substituting $n = +\sqrt{(Ls+R)(Cs+G)}$ (13)

the resulting solutions of (11) and (12) will be of the general form,

$$\overline{I} = A_1 e^{-nx} + B_1 e^{nx} \qquad (45)$$

$$\overline{E} = A_2 e^{-nx} + B_2 e^{nx} \qquad (46)$$

For our problem the constants A and B, which are functions of s but independent of x, will be determined in terms of the generator source resistance, Z_G, the termination resistance Z_T and the characteristic resistance of the line, Z_0. In the more general problem, each of those impedances would be complex, but for our purposes, in order to be able to find solutions for $i = i(x,t)$ and $e = e(x,t)$, we will require that these impedances all be purely resistive. In order to find solutions for (45) and (46), we must also be able to find inverse Laplace transforms of $\overline{E}$ and $\overline{I}$ with the aid of tables in order to obtain e and i as functions of time – then we will have found the desired solution. For quiescent initial conditions, Equation (9) tells us that $\overline{I} = \frac{-1}{Ls+R}\frac{\partial \overline{E}}{\partial x}$ (the term $Li(x,0)$ vanishes) (47)

Therefore, after taking the derivative of (46) to get an expression for $\frac{\partial \overline{E}}{\partial x}$, Equation (45) can be re-written as follows:

$$\overline{I} = \frac{n}{Ls+R}(Ae^{-nx} - Be^{nx}) = \sqrt{\frac{Cs+G}{Ls+R}}(Ae^{-nx} - Be^{nx}) = \frac{1}{Z_0}[Ae^{-nx} - Be^{nx}] \qquad (48)$$

where $Z_0(s) = \sqrt{\frac{Ls+R}{Cs+G}}$

To find expressions for A and B in terms of the termination impedances and the driving e.m.f., we write Kirchoff voltage sums at each of the line. First, at the generator,

$$Z_G(s)\bar{I}(0,s) = F(s) - \bar{E}(0,s) \qquad (50)$$

$$\text{where } F(s) = Laplace\{f(t)\} \qquad (51)$$

Likewise, at the load end,

$$Z_T(s)\bar{I}(d,s) = \bar{E}(d,s) \qquad (52)$$

Substituting the values of Equations (48) and (46) into Eq. (50), we obtain

$$\frac{Z_G(s)}{Z_0(s)}[A(s) - B(s)] = F(s) - [A(s) + B(s)] \qquad (53)$$

and Eq. (52) becomes

$$\frac{Z_T(s)}{Z_0(s)}[A(s)e^{-n(s)d} - B(s)e^{-n(s)d}] = [A(s)e^{-n(s)d} + B(s)e^{-n(s)d}] \qquad (54)$$

Solving Equations (53) and (54) simultaneously for A(s) and B(s),

$$A = \frac{\dfrac{Z_0}{Z_0 + Z_G}}{1 - MNe^{-2nd}} F \qquad (55)$$

$$B = -\frac{\dfrac{Z_0}{Z_0 + Z_G} Ne^{-2nd}}{1 - MNe^{-2nd}} F \qquad (56)$$

Where $M(s) = \dfrac{Z_0 - Z_G}{Z_0 + Z_G}$ and $N(s) = \dfrac{Z_0 - Z_T}{Z_0 + Z_T}$, and F is the Laplace transform of the driving e.m.f.. of the generator

Strictly speaking, M and N are still functions of s at this point. However, later in this document, we will use the following pure resistance values:

- Z_0 = 75 + j0 ohms
- R_G = 225 ohms
- R_T = 25 ohms

Under those conditions, A(s) and B(s) become,

$$A = \frac{0.25}{1 + 0.25e^{-2nd}} F$$

$$B = -\frac{0.125e^{-2nd}}{1 + 0.25e^{-2nd}} F ,$$

which looks very simple, but remember that both n and F are functions of s.

Substituting the expressions for A and B from Equations (55) and (56) into Equations (46) and (48), we obtain

$$\bar{E} = \frac{\frac{Z_0}{Z_0 + Z_G}\left\{e^{-nx} - Ne^{-n(2d-x)}\right\}F}{1 - MNe^{-2nd}} \qquad (57)$$

$$\bar{I} = \frac{\frac{Z_0}{Z_0 + Z_G}\left\{e^{-nx} + Ne^{-n(2d-x)}\right\}F}{1 - MNe^{-2nd}} \qquad (58)$$

Since $|1 - MNe^{-2nd}| < 1$, the denominators of (57) and (58) can be expanded in a power series

$$\bar{E} = \frac{Z_0 F}{Z_0 + Z_G}\left\{e^{-nx} - Ne^{-n(2d-x)} + MNe^{-n(2d+x)} - MN^2 e^{-n(4d-x)} - - -\right\} \qquad (63)$$

At this point it is necessary to disassociate M and N from functions of s and constrain them be pure constants by simply requiring that their reactive components by zero. Therefore, from this point on, the M and N terms are comprised of pure resistance values.

Equation (13) defined n, but since then we have set R and G to zero. Therefore, we can convert the n in Equation (63) to $s\sqrt{LC}$.

$$\bar{E} = \frac{FZ_0}{Z_0 + Z_G}\left\{e^{-s\sqrt{LC}x} - Ne^{-s\sqrt{LC}(2d-x)} + MNe^{-s\sqrt{LC}(2d+x)} + - - -\right\} \qquad (80)$$

Using a table of inverse Laplace transforms,

$$e(x,t) = L^{-1}\left\{F(s)e^{-x\sqrt{LC}s}\right\} = f(t - x\sqrt{LC}).$$

Thus, the voltage at the point x = x is exactly similar to the voltage at the point x = 0 except that there is a time lag of amount $x\sqrt{LC}$.

Applying the inverse transformation to each term of (80) gives us the solution, as follows, in which the general function of time, f(t) must still be specified, and can be any form of excitation desired, such as a step function, a Dirac function, or a simple harmonic function of time.

$$e(x,t) = \frac{Z_0}{Z_0 + Z_G)} \{ f(t - x\sqrt{LC}) - Nf[t - (2d - x)\sqrt{LC}]$$

$$+ MNf[t - (2d + x)\sqrt{LC}] + --- \} \qquad (81)$$

It is clear from Equation (81) that the voltage at the generator terminals will consist of a infinite series of terms in which the odd numbered terms are incident waves traveling toward the load and the even numbered terms are reflected waves traveling toward the source, and that the constant coefficients, M and N, associated with each term are simply the classic reflection coefficients that we use in the steady-state analysis.

With a simple harmonic function of time, $V\sin(\omega t)$, as the driving function the terms of Equation (81) become:

$$\begin{aligned} e(x,t) = \frac{VZ_0}{R_G + Z_0} \{ & \sin(\omega(t - x\sqrt{LC})) \ - N\sin(\omega(t - (2d - x)\sqrt{LC})) \\ & + MN\sin(\omega(t - (2d + x)\sqrt{LC})) \ - MN^2\sin(\omega(t - (4d - x)\sqrt{LC})) \\ & + M^2N^2\sin(\omega(t - (4d + x)\sqrt{LC}))......... \ \} \end{aligned}$$

where each term is seen to have a delay corresponding to the time spent traveling from the generator to point x, including any trips to the load and back, as appropriate for that term.

In order to understand how we get from Equation (81) to the equation above, remember that each term in the result is simply the inverse Laplace transform of its corresponding term in Equation (81).

It is important to realize that $e(x,t)$ in the above equation is not a phasor. The phasor (complex exponential form of voltages and currents) is valid only in the steady-state and has no mathematical basis for use in transient analyses (which is one of the reasons why so many attempts at transient analysis are seriously flawed).

The expression $\frac{1}{\sqrt{LC}}$ is the signal velocity (or group velocity – see page 290), and the expression $\sqrt{LC}$ has the units of seconds per unit length. For a given radian frequency ω, wavelength λ, and velocity factor for the line (VF), the following expression can be substituted for $\sqrt{LC}$:

$$K = \frac{2\pi}{\omega\lambda(VF)},$$

which represents the reciprocal of the actual velocity and has the units of seconds per unit length. Therefore, we can write the above equations as follows:

$$e(x,t) = \frac{VZ_0}{R_G + Z_0}\{\sin(\omega(t - xK)) \ - N\sin(\omega(t-(2d-x)K))$$
$$+ MN\sin(\omega(t-(2d+x)K)) \ - MN^2\sin(\omega(t-(4d-x)K))$$
$$+ M^2N^2\sin(\omega(t-(4d+x)K)).........\}$$

Note that for a given d and x, the term $(nd + x)K$, when multiplied by ω, gives the total phase delay (radians) relative to the first term.

By assigning values to the impedances, length of the line and propagation velocity we can obtain numerical results for a specific point of observation on the line. The following impedances, dimensions and other values will be used for purposes of illustration:

- Z_0 = 75 + j0 ohms
- R_G = 225 ohms
- R_T = 25 ohms
- ω = 62.8 * 10^6 radians/sec (f = 10 MHz)
- d = 7.5 meters (1/4 wavelength at 10 MHz)
- x = 0 (point of observation = generator terminals)

From the above configuration parameters, we can also calculate values for K, M and N, as follows:

- K = 10^{-6}/300 = .003333 * 10^{-6} seconds/meter (VF = 1.0)
- M = -0.5
- N = 0.5

The instantaneous voltage at the generator terminals (x = 0) for any time, t, is as follows:

$$e(t) = \frac{V}{4}\{\sin(\omega \cdot t) \ - 0.5\sin(\omega \cdot t - \pi)$$
$$- 0.25\sin(\omega \cdot t - \pi) \ + 0.125\sin(\omega \cdot t - 2\pi)$$
$$+ 0.0625\sin(\omega \cdot t - 2\pi).........\}$$

The above series shows clearly that the incident wave (the first term) becomes a reflected wave after traveling the length of the line. Upon reaching the generator end of the line, that reflected wave (the second term) then becomes a weaker incident wave (the third term), and so on, forever.
Clearly, the phase delay of -2π (360 degrees) in the 4th and 5th terms have no effect on the numeric solution and can be dropped. However, the phase delay of $-\pi$ in the 2nd and 3rd terms introduces a (-1) factor, which changes the sign of the 2nd and 3rd terms, as follows:

$$e(t) = \frac{V}{4}\{\sin(\omega \cdot t) \ + 0.5\sin(\omega \cdot t)$$
$$+ 0.25\sin(\omega \cdot t) + 0.125\sin(\omega \cdot t)$$
$$+ 0.0625\sin(\omega \cdot t).........\}$$

After an infinite number of terms have been summed, the steady-state solution results, as follows:

$$e(t) = \frac{V}{4}\{\sin(\omega \cdot t) + 0.5\sin(\omega \cdot t).........\} = \frac{V}{2}\sin(\omega \cdot t)$$

In order to better understand the early transient phase it is necessary to consider the reflections arriving at the generator end of the line as discrete events. The conditions prior to and immediately following the arrival of a given reflection indicates that a step change of voltage and current occurs, which immediately affects the impedance seen looking into the line at that point.

Before the first echo arrives, the voltage as a function of time is one half of the steady-state value. After the first echo arrives, the voltage at the generator terminals is $\frac{1.5V}{4}\sin(\omega \cdot t)$. After the arrival of the second echo, the voltage at the generator terminals becomes $\frac{1.75V}{4}\sin(\omega \cdot t)$, and so on, forever.

The equation for the instantaneous current may be derived in similar fashion, as follows:

$$\begin{aligned} i(x,t) = \frac{V}{R_G + Z_0}\{&\sin(\omega(t - xK)) + N\sin(\omega(t-(2d-x)K)) \\ &+ MN\sin(\omega(t-(2d+x)K)) + MN^2\sin(\omega(t-(4d-x)K)) \\ &+ M^2N^2\sin(\omega(t-(4d+x)K))\text{.........}\ \} \end{aligned}$$

After substituting in the numerical values as were used in the voltage equation, the current as a function of time is as follows:

$$\begin{aligned} i(t) = \frac{V}{300}\{&\sin(\omega \cdot t) + 0.5\sin(\omega \cdot t - \pi) \\ &- 0.25\sin(\omega \cdot t - \pi) - 0.125\sin(\omega \cdot t - 2\pi) \\ &+ 0.0625\sin(\omega \cdot t - 2\pi)\text{.........}\ \} \end{aligned}$$

After taking into account the effect of the phase delay terms, which reverse the signs of the second and third terms, we obtain:

$$\begin{aligned} i(t) = \frac{V}{300}\{&\sin(\omega \cdot t) - 0.5\sin(\omega \cdot t) \\ &+ 0.25\sin(\omega \cdot t) - 0.125\sin(\omega \cdot t) \\ &+ 0.0625\sin(\omega \cdot t)\text{.........}\ \} \end{aligned}$$

It can be shown that the steady-state expression for current is as follows:

$I = \frac{V}{450}$ amps peak (the generator sees a matched load of 225 ohms).

Prior to the arrival of the first echo, the incident current wave will be as follows:

$$i(t) = \frac{V}{300}\sin(\omega \cdot t).$$

Since the voltage during that same period is $e(t) = \frac{V}{4}\sin(\omega \cdot t)$, the instantaneous impedance seen at the generator terminals is $\frac{300}{4} = 75$ (the concept of "instantaneous impedance" will be discussed at the end of this paper).

Upon the arrival of the first echo, the total current becomes

$$i(t) = \frac{V}{300}\{\ \sin(\omega \cdot t) - 0.5\sin(\omega \cdot t)\ \} = \frac{V}{600}\sin(\omega \cdot t).$$

At the same time, the total voltage becomes $\frac{1.5V}{4}\sin(\omega \cdot t)$.

We will refer to the quotient of the voltage divided by the current (225 ohms) as the instantaneous impedance. Thus, the arrival of the first echo presents a perfect match to the generator.

The following table gives voltage, current and *instantaneous impedance* for 1st term only, the sum of the 1st and 2nd terms, then the sum of the 1st, 2nd and 3rd terms, etc:

	Voltage	Current	Instantaneous Impedance (ohms)
1st Term only	$\frac{V}{4}\sin(\omega \cdot t)$	$\frac{V}{300}\sin(\omega \cdot t)$	75
Incl. 2nd term	$\frac{1.5V}{4}\sin(\omega \cdot t)$	$\frac{0.5V}{300}\sin(\omega \cdot t)$	225
Incl. 3rd term	$\frac{1.75V}{4}\sin(\omega \cdot t)$	$\frac{0.75V}{300}\sin(\omega \cdot t)$	175
Incl. 4th term	$\frac{1.875V}{4}\sin(\omega \cdot t)$	$\frac{0.625V}{300}\sin(\omega \cdot t)$	225
Incl. 5th term	$\frac{1.9375V}{4}\sin(\omega \cdot t)$	$\frac{0.6875V}{300}\sin(\omega \cdot t)$	211
…	…	…	…
Steady-state	$\frac{V}{2}\sin(\omega \cdot t)$	$\frac{V}{450}\sin(\omega \cdot t)$	225

It is interesting to compare the *instantaneous impedance* results in the above table with similar calculations using Goldman's steady-state equations in his Appendix C. In the steady-state solution, which is also an infinite power series, the impedances indicated when summing the first 5 terms of the series are identical with the corresponding *instantaneous impedances* in the table above. Goldman introduces his steady-state analysis of transmission lines by stating that the analysis will provide a "good physical picture of what is going on" in steady-state operation. As his analysis progresses, he shows that those energies are actually the superposition of two traveling waves – one traveling toward the load and the other traveling away from the load. Finally, he expands those two wave expressions into a convergent power series which shows that there are not two traveling waves, but rather an infinite number of individual multiple reflections.

The steady-state analysis differs from the transient analysis above in one respect only - it uses instantaneous complex exponential representations of voltage and current (phasors) instead of instantaneous functions v(t) and i(t).

One might question the concept of "instantaneous" impedance calculated from the instantaneous functions $e(t)$ and $i(t)$ inasmuch as the concept of "impedance" is normally restricted to steady-state conditions in which phasor (rotating vector) notation is used.

The functions $e(t)$ and $i(t)$ are not phasors, so it is incorrect to characterize the ratio of the instantaneous voltage to the instantaneous current as “impedance” (that would be valid only if the voltages and currents were simple harmonic functions of time under steady-state conditions). In the steady-state there would be no question about the validity of characterizing the impedance at a point on the line as the ratio of the summed voltages to the summed currents existing at that same point, taking into account the phase angles involved. Nonetheless, having warned the reader, we introduce the concept of “instantaneous” impedance in order to illustrate the similarity between the equations of the transient phase and the steady-state equations. For purposes of this discussion, we presume that the ratio of summed voltages to summed currents at a point on the line, taking into account the phase angles, gives an impedance – just as if the voltages and currents were true phasors.

In conclusion, the Goldman equations provide valuable insight into the nature of the incident or forward (odd numbered) and reflected (even numbered) transient waves on the transmission line. They are especially revealing in regard to the rapidity with which the higher numbered terms become vanishingly small. The total contribution of all terms after the first five is less than 0.1% of the steady-state power level.