Understanding the Goldman Equations for the Steady-State Analysis as They Apply to the Quarter Wave Matching Transformer

[Stanford Goldman's "Transformation Calculus and Electrical Transients"]

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Introduction

In 1949, Stanford Goldman, Professor of Electrical Engineering at Syracuse University, wrote his book, "Transformation Calculus and Electrical Transients", which was published by Prentice-Hall, Incorporated as part of their Electrical Engineering Series, edited by W. L. Everitt. I was given a copy of this book by my late friend and colleague, Sukru Cafer Durusel, who died in July 1965. My interest in the problem of explaining where the reflected power goes began in 1995 when my interest in RF transmission lines was re-awakened after returning to Amateur Radio after an absence of 25 years. I found that the equations provided in the ARRL Antenna Handbook and in the ITT Handbook for Radio Engineers were more than adequate to explain the impedance seen at any point on a transmission line, given its characteristic impedance and its terminating impedance. I also found that there was considerable disagreement in the community on certain aspects of the theory of standing waves. In particular, the condition in which a transmission line has an impedance discontinuity at both the source end and the load end. The standing wave pattern on such a line poses an enigma in that the reflected wave energy appears to simply stop and vanish at the source end of the line. There have been innumerable theories and explanations posed as to where that power goes, and no theory appears to satisfy everyone. I have written two articles, one addressing the problem in the transient phase and the other addressing the problem in the steady-state. I found that Goldman's work on transmission lines is the only published work that covers both aspects of the problem and provides a unified mathematical basis for each. The objective of these two articles is to try to bring the mathematical concepts outlined by Goldman to bear on specific, well known examples of the quarter wave matching transformer. The reader will find that no matter whether the transient or the steady-state model is pursued, the Goldman equations will provide a clearer picture of what really goes on in the transmission line. Each article begins with an explanation of the mathematical basis, which is then followed by a numerical example for a specific quarter wave matching transformer.

This paper addresses Appendix C of the subject book – in particular, the steady-state analysis of a terminated transmission line.

Figure 10-4 shows the simple transmission line circuit with its source generator and a complex load. The objective is to provide an analysis of the instantaneous voltage and current as a function of time for any distance "x" along the transmission line of length "D" under steady-state conditions. Our discussion will apply directly to a quarter wave matching transformer, where $D = \lambda / 4$.

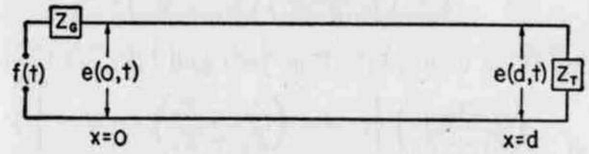

FIG. 10-4. A finite line with terminal impedances.

The fundamental equations which govern phenomena in a transmission line are

$$L\frac{\partial i}{\partial t}+Ri=-\frac{\partial v}{\partial x} \qquad (8)$$

$$C\frac{\partial v}{\partial t}+Gv=-\frac{\partial i}{\partial x} \qquad (9)$$

The two partial differential equations are an expression of Kirchoff's law for an infinitesimal length of line having series inductance of L Henries per meter, series resistance of R ohms per meter, shunt capacitance of C Farads per meter and conductance G mhos per meter. Equation (1) says that the Kirchoff voltage loop consists of the self induced voltage in volts per meter, the series resistance drop in volts per meter and the total voltage change per meter. Likewise Equation (2) says that the Kirchoff current summation consists of the capacitive charging current in amperes per meter, the conductance or leakage current in amperes per meter and the total shunt current per meter.

Note that the observation point, x, is the distance from the generator terminals in the direction of the load.

In Appendix C of the subject textbook, Goldman first develops the steady-state input voltage and current in terms of an output voltage and current at the load, a propagation constant for the line n, a length of line D, and a characteristic impedance of the line Zo. He further develops an expression for the input impedance of the line in terms of the length of the line, the propagation constant of the line, the characteristic impedance of the line and the terminating impedance. The aforementioned relationships are all well known for transmission lines in the steady-state.

He then points out that these relationships do not give a clear picture of what is going on. Using simple hyperbolic identities, he re-arranges the well know relationships into a complex exponential form that illustrates clearly that both the current and the voltage on the line consist of the superposition of two traveling attenuated cosine waves – one of which is traveling in the direction of the load and the other which is traveling in the direction of the source.

Again, the aforementioned concept of forward and reflected waves is well known.

Goldman then presents an analysis in terms of multiple reflections which is accomplished through another set of manipulations resulting in a solution in the form of a convergent power series. That solution is as shown in the following equations:

$$i = \frac{E}{Z_0 + Z_G} \{\epsilon^{-\alpha x}\epsilon^{j(\omega t - \beta x)} + N\epsilon^{-\alpha(2D-x)}\epsilon^{j[\omega t - \beta(2D-x)]}$$
$$+ MN\epsilon^{-\alpha(2D+x)}\epsilon^{j[\omega t - \beta(2D+x)]} + MN^2\epsilon^{-\alpha(4D-x)}\epsilon^{j[\omega t - \beta(4D-x)]}$$
$$+ M^2N^2\epsilon^{-\alpha(4D+x)}\epsilon^{j[\omega t - \beta(4D+x)]} + M^2N^3\epsilon^{-\alpha(6D-x)}\epsilon^{j[\omega t - \beta(6D-x)]}$$
$$+ \text{---}\} \quad (84)$$

$$v = E\frac{Z_0}{Z_0 + Z_G} \{\epsilon^{-\alpha x}\epsilon^{j(\omega t - \beta x)} - N\epsilon^{-\alpha(2D-x)}\epsilon^{j[\omega t - \beta(2D-x)]}$$
$$+ MN\epsilon^{-\alpha(2D+x)}\epsilon^{j[\omega t - \beta(2D+x)]} - MN^2\epsilon^{-\alpha(4D-x)}\epsilon^{j[\omega t - \beta(4D-x)]}$$
$$+ M^2N^2\epsilon^{-\alpha(4D+x)}\epsilon^{j[\omega t - \beta(4D+x)]} - M^2N^3\epsilon^{-\alpha(6D-x)}\epsilon^{j[\omega t - \beta(6D-x)]}$$
$$+ \text{---}\} \quad (85)$$

An understanding of Equations (84) and (85) may lead to an even clearer physical picture of what is really going on in a transmission line. Furthermore, we will find, upon comparing this approach with his transient analysis in Chapter 10, that his steady-state equations and his transient equations present a unified and consistent result in terms of an infinite number of multiple reflections.

Goldman's equations for the steady state provide the instantaneous complex voltage and complex current at point x on a transmission line of length D, characteristic impedance Z_0, terminated by a load Z_T and driven by a generator of internal impedance Z_G. These equations are in the form of an infinite, convergent power series. Goldman's coefficients M and N are simply the negative of the usual reflection coefficient for the source mismatch and the load mismatch, respectively.

The first and all odd numbered terms in each of these equations are forward (incident) voltage and current terms. That is, they are traveling in the direction of the load. The second and all even numbered terms in each of these equations are the reflected voltage or current terms, and they are traveling toward the source. The individual odd numbered terms, when added as complex quantities, sum to the value of the complex forward (incident) voltage or current as would be measured with an SWR meter as incident or forward power. The individual even numbered terms, when added as complex quantities, sum to the value of the complex reflected voltage or current as would be indicated with an SWR meter as reflected power. Note that an SWR meter does not measure the individual terms in each direction but measures the algebraic sum of all terms in a given direction.

The terms taken individually represent energies that are of different ages in the transmission line. For example, the 1st terms are of the current age and have suffered the reduction of transmission through one discontinuity, while the 2nd terms are of the previous age - that is, they have also suffered a reduction through one reflection. The 3rd terms are of yet one earlier age – that is, they have suffered a re-reflection. Each successive term is again reduced in magnitude in comparison with its predecessor by virtue of the additional reflection coefficient that it has encountered.

For purposes of discussion we will present numeric data based upon the following system:

- Z_0 = 75 + j0 ohms
- R_G = 225 ohms
- R_T = 25 ohms
- ω = 62.8 * 10^6 radians/sec (f = 10 MHz)
- D = 7.5 meters (1/4 wavelength at 10 MHz)
- x = 0 (point of observation = generator terminals)

For a 75 ohm quarter wave transformer section terminated in a purely resistive 3:1 mismatch, the applicable reflection coefficient is 0.5, thereby resulting in a reduction of 2:1 from one term to the next. As a consequence of the voltage and current each being reduced 2:1 on each reflection, the power in each pair of voltage and current terms is reduced by the factor 4:1, as can be seen in the following table: (Note that all voltages and currents are peak, not r.m.s., at point x.)

1^{st} Term	106.05v	1.414a	75w
2^{nd} Term	53v	.707a	18.75w
3^{rd} Term	26.5v	.353a	4.7w
4^{th} Term	13.25v	.177a	1.7w
5^{th} Term	6.6v	.088a	.29w
6^{th} Term	3.3v	.044a	.073w
7^{th} Term	1.6v	.022a	.018w
8^{th} Term	0.83v	.011a	.0046w
…	…	…	…
Totals	212v	.943a	100w

The significance of this data is that it illustrates that at any given moment, the entire history of the multiple reflections exists in the form of an infinite number of ever smaller, older components whose origins are in the original start up transient phase. From a practical viewpoint, it is also obvious that the energy components are vanishingly small after the 8^{th} term.

It should be equally obvious that as a consequence of this being a steady state solution, the total amount of energy in the sum of all terms (100 Joules/sec) equals the total amount of energy being delivered continuously to the load, which is also being balanced continuously by an equal amount of new energy from the source. The overall effect of this is that there is a steady-state balance between energy flows into and out of the quarter wave transmission line.

If we sum the odd numbered voltages and the odd numbered currents from the terms in the Goldman equations, we obtain a forward voltage of 100 volts r.m.s. and a forward current of 1.333 amps r.m.s., giving a forward power of 133.33 watts.

Likewise, if we sum the even numbered voltages and the even numbered currents from the terms in the Goldman equations, we obtain a reflected voltage of 50 volts r.m.s. and a reflected current of 0.666 amps r.m.s., giving a reflected power of 33.33 watts. The net power level in the quarter wave transformer is therefore, 100 watts - displayed as the total power in the above table.

If we examine the ratio of any one of the voltages to its corresponding current, we find that it equals 75 ohms for any one of the individual terms and that ratio is 225 ohms for the summed voltage summed current. In theory, the summed voltage and current are comprised of an infinite number of terms. In practice, the first 40 terms are more than adequate to account of 99.99% of the energy. These values are consistent with the characteristic impedance of the quarter wave line and with the impedance seen looking into that line, as it is configured. In other words, everything about the operating parameters of the line, as determined from the Goldman equations, is 100% consistent with what we know about the line from other analytical methods.

Let us now examine these equations for information about the amount of energy stored in the line, how much energy is being replenished in a given interval by the source and how much is being given up by the line to the load in that same interval. We know that the line is of ¼ wavelength, and we know the time it takes for a wave to travel that distance is 1/4F microseconds when the frequency, F, is in MegaHerz. For example, at 10 MHz the wave will travel the length of the line in .025 microseconds.

Since the sum total of all energy flows from the infinite number of terms in the Goldman equations is known to be 133.33 Joules per second (watts) in the forward direction and 33.33 Joules per second (watts) in the reverse or backwards direction, then the grand total of the energy present in the line in the form of traveling waves is 166.66 Joules per second times .025 microseconds or 4.1666 microJoules.

The question will certainly arise as to why the two energy flows combine to 166.66 instead of 100 as one might expect, considering that they are flows in opposite directions. The analogy is simple. If there are 133 automobiles going north on a given highway, while 33 automobiles are traveling south on that same highway during a given interval, then how many automobiles will be counted passing a given checkpoint during that interval? The answer, of course, is 166 – not 100.

Next is the question of how many Joules are provided to the line in each .025 microsecond interval by the source, and the answer is 100 Joules/sec (watts) times the 0.025 microsecond interval or 2.5 microJoules.

Finally, the remaining question is, how many Joules are supplied by the line to the load in a given interval? Since the load is consuming energy at the rate of 100 Joules/sec (watts), then the load consumes 2.5 microJoules in the .05 microsecond interval – the same amount as delivered to the line by the source. Therefore, we can conclude that approximately 60% of the energy stored in the line is being given up to the load in any given interval and that the source replenishes an equal amount of energy at that same rate. It is important to remember that these values are for the particular configuration of line and frequency as declared above. Other lines operating at different lengths and frequencies may yield different values.

Are there any particular advantages or disadvantages in the use of the Goldman steady-state model as compared with the classic transient analysis? In the transient analysis, at the end of each half wave interval or round trip, the resulting wave energies are computed and combined to form a new set of initial forward and reflected terms to beginning the next interval. All important values are recorded for that step and the next step is then taken. This process continues until the values are essentially constant (steady-state). At that point the energy that was being tracked has reflected back and forth over the quarter wave section, losing a portion to the load and losing a portion to the input line in each interval, to the point where it has essentially decayed to zero.

The record of the transient analysis is the record of the change in state of the line at the end of each interval. During each interval, the amount of energy remaining from that original amount entering the line is replenished with new energy entering during that same interval so as to form an ever increasing amount of total energy in circulation around the quarter wave section. Therefore, when steady-state values are finally attained, we have a history of the state at the end of each interval, and we have the steady-state values.

In contrast with that, the Goldman steady-state equations provide us with a fixed set of values of an infinite number of terms making up the steady-state solution. The terms in the steady-state solution can be combined and analyzed in many different ways but the data itself is fixed or unchanging with time. Each term in the Goldman steady-state equations has been present forever.

Is there any significant value in studying the Goldman steady-state equations in addition to the typical transient analysis?

To answer that, we must review what the Transient Analysis diagram tells us and ask ourselves what it provides. First it requires that we analyze the decay of energy due to each reflection that occurs. Finally, it leaves behind a step-by-step history of how the original wave energy has decayed as it reflects and re-reflects at each discontinuity, shedding a portion of its energy at each reflection. An interesting side issue is the correlation between the numerical results in the steady-state equations and the numerical results from the transient analysis equations. For a given hardware configuration and a given source e.m.f., we find that the terms are numerically equal.

How do the overall results of a Goldman transient analysis compare with the information from the Goldman steady-state equations? The Goldman steady-state equations are not a step by step transient analysis starting at t = 0. Instead, they provide a solution for the complex voltage and current on the line at steady-state as an infinite, convergent power series of complex voltages and currents, each term of which represents separate and distinct wave energy eternally present on the line. If we look at the sum of all the terms we see only the steady state solution, but the history of the transient phase becomes apparent when you look at the individual terms in that series – especially, when you consider that they have exactly the same numerical values for identical configurations.

In comparing this information with information from a Transient Analysis diagram and the corresponding analytical discussion of the transient phase, we find that there is information in the Goldman equations that we can use to independently arrive at the same end result as the transient analysis. For example, we find that when we sum the odd numbered voltage terms and multiply that sum by the sum of the odd numbered current terms, we obtain the forward power in the line (141.4 x 1.885/2 = 133.33 watts). (The voltages and currents are peak rather than r.m.s. values – therefore, the product of peak voltage and peak current must be divided by 2 in order to obtain watts). Likewise, if we multiply the sum of the even numbered voltages by the sum of the even numbered currents, we obtain the reflected power in the line (70.7 x 0.944/ = 33.33 watts). We can now begin to see several unique aspects of the Goldman steady-state equations, as follows:

- The infinite series of terms is clearly comprised of both the forward and reflected waves.
- The total current or total voltage at any point on the line is the superposition of an initial wave (first term) plus its reflection at the load (2^{nd} term) plus the reflection of that wave at the generator (3^{rd} term) plus multiple reflections back and forth from the ends of the line.
- Each term can be directly associated with a specific number of trips along the line.

- The age of each term since power was first applied can be determined from its number.
- The power level defined from a given voltage and current term can be combined algebraically with power levels from other terms to determine the total forward power level, the total return power level and the net power level.
- The complexity of the infinite number of co-existing energies on the line is revealed – i.e., we can visualize aspects of the individual wave energies that are not directly visible in the step-by-step transient analysis.

Therefore, the ability to visualize the true nature of the energy flows in the line is enhanced by the Goldman steady-state equations and is a valuable complement to the step-by-step transient analysis. Above all else, it shows that there is not just one wave that is decaying on the line at any given moment, but rather that there is an infinite number of such waves decaying on the line, continuously, and that those waves are being replenished with new wave energy at the same rate as they give up their energies at the discontinuities.