Demonstrating Celestial Mechanics Through Measured Doppler Shift

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ABSTRACT

Stable orbits (of natural and artificial satellites alike) require an equilibrium between gravitational and inertial forces. For a given satellite altitude, or orbital radius, there is but a single orbital velocity which affords such an equilibrium state. This paper shows how students at the Pennsylvania College of Technology utilize Doppler shift measurements of received satellite telemetry signals to accurately determine the orbital period of Amateur Radio communications satellites, and from it, their other orbital parameters.

FUNDAMENTAL ORBITAL MECHANICS

A stable orbit, whether of a satellite around a planet or a planet around a sun, requires that the inward pull of gravity and the outward pull of inertia be equal. Kepler tells us that all satellites orbit their primaries in an ellipse, and that orbital velocity changes throughout the elliptical orbit (fastest at perigee, slowest at apogee) in order to maintain equilibrium. For the present study, we will restrict ourselves to analyzing the behavior of satellites in roughly circular orbits (that is, orbital eccentricities near zero), so that the satellite's orbital velocity is essentially constant. Fortunately, the current generation of MicroSats fills the bill almost perfectly.

Newton's famous Inverse Square Law shows the force of the gravitational attraction between any two bodies to equal a fudge factor (the Universal Gravitational Constant), times the product of their masses, divided by the square of the distance between their centers of mass. Mathematically,

$$F = \frac{GMm}{r^2} \qquad \text{[Equation 1]}$$

where

M is the mass of the primary (in our case planet),
m is the mass of the secondary (satellite),
r is the distance between them (the orbital radius),
and
$G = 6.673 \times 10^{-11}$ Nt m^2 / kg^2, Newton's Universal Gravitational Constant.

We now consider the force of inertia pulling a satellite out, which (again according to Newton) equals:

$$F = mA \qquad \text{[Equation 2]}$$

where
m represents the mass of the satellite,
and
A is its acceleration, which in a circular orbit is found from:

$$A = \frac{v^2}{r} \qquad \text{[Equation 3]}$$

with
v representing the velocity of the satellite,
and
r the orbital radius, as defined above.

Combining Equations 2 and 3 gives us:

$$F = \frac{mv^2}{r} \qquad \text{[Equation 4]}$$

from which we could determine the inertial force acting on the satellite, given its mass and orbital velocity. Velocity is of course related to orbital period, which we will derive shortly from Doppler shift measurements.

Since our Amateur Radio satellites appear (thankfully) to be in stable orbits, we set Equations 1 and 4 equal to each other:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \qquad \text{[Equation 5]}$$

and then simplify:

$$\frac{GM}{r} = v^2 \qquad \text{[Equation 6]}$$

We can now solve Equation 6 for v:

$$v = \left(\frac{GM}{r}\right)^{1/2} \qquad \text{[Equation 7]}$$

or for r:

$$r = \frac{GM}{v^2} \qquad \text{[Equation 8]}$$

and we see that the velocity and orbital radius of our satellite are inexorably linked, by readily determined constants.

DETERMINING THE GM PRODUCT

At the Pennsylvania College of Technology, second year electronics students have recently come up with an independent estimate of the Earth's mass, based upon recovering echoes from radio signals bounced off the surface of the Moon. Their novel EME experiment, which involved observing the lunar orbit and solving Equation 6 above for M, has already been treated in the literature [Shuch, 1991]. Their published result for the mass of the Earth, 6.037×10^{24} kg, appears to be in error by about 1%.

Let's utilize a more widely accepted value for the mass of the Earth: 5.975×10^{24} kg. Now we've already stated that Newton's Universal Gravitational Constant, a fudge-factor for dimensional consistency, is equal to 6.673×10^{-11} Nt m^2 / kg^2. Thus we see that the GM product encountered in Equations 7 and 8 above is not a Chevy at all, but rather 4×10^{14} m^3/s^2, a constant which relates radius to velocity for any satellite orbiting the Earth.

DOPPLER, AND OTHER SHIFTY CHARACTERS

The change in frequency of electromagnetic waves as a function of relative motion is now known as the Doppler shift. The phenomenon was first described by Johann Christian Doppler, a mathematics professor at the State Technical Academy in Prague, in 1842, in a paper delivered to the Royal Bohemian Society of Learning titled "On the Colored Light of Double Stars and Some Other Heavenly Bodies" (Magnin, 1986). Doppler shift varies directly with both the transmitted frequency and the relative velocity between the transmitter and receiver, and inversely with the speed of light. It is utilized in fields as diverse as aircraft radar (Shuch, 1987), spacecraft navigation, remote sensing, biomedical imaging, and of course satellite orbital analysis (Davidoff, 1978).

To understand the Doppler shift for electromagnetic waves, imagine the headlight on the front of an approaching train, which is traveling at a substantial velocity—let's say, mach 100,000, a tenth the speed of light. Now we know the radiation leaves the headlight at the speed of light, 3×10^8 meters per second. Since it appears that the train is adding its forward velocity to that of the light beam, we would naively expect the light from the moving train to reach us at a speed 10% greater than that at which it left the bulb, or 3.3×10^8 m/s.

But of course it can't. Einstein tells us that the speed of light in free space, whether measured at the point of transmission, the point of reception, or some point in between, will *always* equal exactly the same value, 300 million meters per second. The wave *cannot* change speed, regardless of relative motion between the observers. Yet the presumed additional velocity which we had expected the train's motion to impart to the wave has to go somewhere. And since it can't manifest itself in a speed variation, it instead varies the *frequency* of the wave.

The Doppler shift is remarkably symmetrical. It cares not whether the source of relative motion is the transmitter, the receiver, or some combination of the two. And the magnitude of the frequency shift is the same whether the length of the transmission path is increasing or decreasing, though of course its direction varies. Moving closer together, blue shift, increasing frequency. And moving farther apart, red shift, a decrease in frequency.

Radio amateurs have been aware of the Doppler shift within the context of space communications, ever since they began bouncing signals off the surface of the Moon nearly forty years ago. As the Moon is rising, moving toward us (or more properly, as we are rotating toward it), our echoes come back higher in frequency than the transmitted signal. The setting Moon (moving away from us, or more properly us away from it) gives us the opposite effect, down Doppler, decreasing frequency.

The phenomenon was spectacularly evident to those space communications pioneers who first recovered Sputnik I's 20-MHz beeps on October 4, 1957.[1] However, the Sputnik signals had so much chirp on them that more than one observer overlooked the Doppler shift as yet another manifestation of an instable transmitter. Today we often design the transponders of communications satellites with frequency inverting passbands, in an effort to partially cancel this ever-present "designed-in drift."

The easiest way to quantify the Doppler shift is to think of it as a simple ratio. The Doppler change in frequency f_d, is to the transmitted frequency (f_o) as the relative velocity (v) is to the velocity of the transmitted wave (which we know to equal c, the speed of light). We formalize this relationship as:

$$\frac{f_d}{f_o} = \frac{v}{c} \qquad \text{[Equation 9]}$$

The equation can also be solved for the relative velocity between the points of transmission and reception:

$$v = \frac{c \times f_d}{f_o} \qquad \text{[Equation 10]}$$

[1]Notes appear on page 7

which will enable us to determine the orbital velocity of a communications satellite, from the maximum Doppler shift observed on its telemetry beacon, or other transmitted signal.

SATELLITE SLEUTHING

This avocation has been raised to the level of high art-form by Geoff Perry and others of the legendary Kettering Group in England, and the techniques discussed here should certainly be attributed to them (Davidoff, 1990, 14-12 to 14-17). The key to determining the orbital characteristics of an "unknown" satellite is to observe Doppler-induced changes in its apparent frequency, and to graph them over time. If we can accurately observe Time of Closest Approach (TCA), along with Acquisition of Signal (AOS) and Loss of Signal (LOS) times, then we can estimate the satellite's orbital period. From that we can compute altitude and velocity, thence estimate AOS, LOS and TCA for future orbits.

The dedicated satellite sleuth relies upon not only direct observation, but past experience in determining orbital parameters. A thorough database of the characteristics of known satellites is built up, to which a newcomer can be compared in trying to determine its general orbit, and speculate as to its mission.

SELECTING A SATELLITE

The true satellite sleuth delights in "discovering" new satellites, and working out as many of their orbital characteristics as possible, armed with little more than a receiver with which to recover their signals. The purpose of the present exercise is somewhat different: to demonstrate to the student the relationships between the orbital parameters of a satellite, and to illustrate how a balance of forces defines the orbit. Thus a truly "unknown" satellite is hardly a requisite. In fact, the exercise has even more instructional validity if the measurements are made on a satellite of known orbital characteristics, against which the student's results can be compared. Let us consider, for example, analyzing the 70-cm CW signals from the LUSAT-OSCAR 19 (MicroSat D).

This particular signal is chosen for my students' first exercise in orbital analysis for a number of reasons. The 437.127-MHz frequency is high enough to provide ample, easily observed Doppler shift (remember, f_d varies directly with frequency). The 750-mW beacon signal is strong enough to be readily received on relatively simple equipment. CW is the preferred modulation mode for accurate Doppler measurements, because the signal can be zero-beat on a receiver with direct digital frequency readout. Finally (and this is cheating), the orbits of low-altitude, circular, sun-synchronous, near-polar satellites such as all four of the 1990 MicroSats are especially well suited to the type of measurements required. In other words, if you pick a satellite with the right orbital parameters, it's easy to determine its orbital parameters![2]

CONDUCTING THE EXPERIMENT

We begin much as the satellite sleuth begins, measuring Doppler shift over successive orbits and displaying it graphically. The procedure, well documented in the literature (Talcott Mountain Science Center, 1975), is repeated here for the benefit of those who might not have seen it in its entirety.

Once able to successfully (and consistently) receive the telemetry beacon from OSCAR 19, the student is asked simply to measure, as accurately as possible, the received signal frequency, at one minute intervals all the way from AOS to LOS. This is done initially for two successive orbits. The TCA of the satellite to the observer is indicated by the maximum slope of the plotted Doppler curves, as illustrated in Figs 1 and 2. These are of course the familiar Doppler S-curves, which we've used since the days of OSCAR VI. Their continuously varying slope (rate of change, or first derivative) holds the key to evaluating the satellite's orbit.

The time difference between two successive TCAs is a first order approximation of the satellite's orbital period. It is *only* an approximation, since the effect we are actually measuring involves not only the satellite's orbital motion, but also the eastward rotation of the Earth. For a more precise measurement, we determine the elapsed

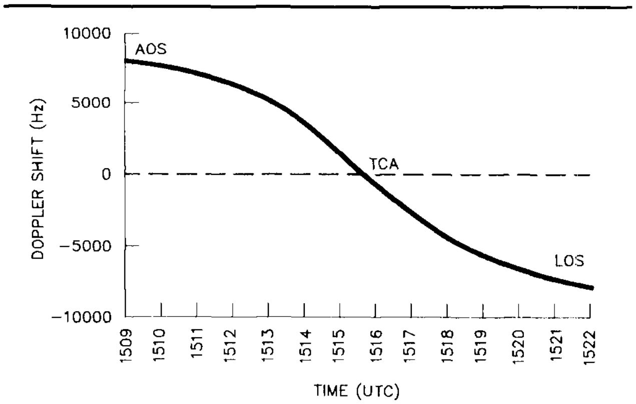

Figure 1—LO-19, 22 Aug 91

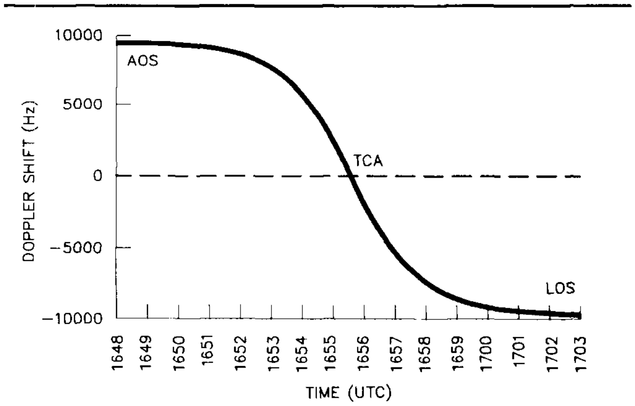

Figure 2—LO-19, 22 Aug 91

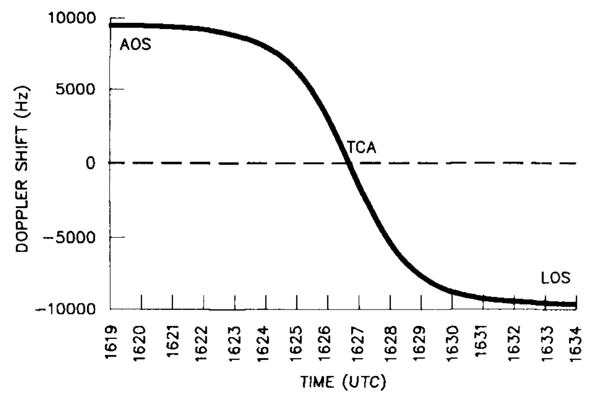

Figure 3—LO-19, 23 Aug 91

time between successive overhead passes. If the satellite is monitored for an extended period, eventually an orbit is encountered which closely approximates a direct overhead pass. This is evidenced by a maximum period of visibility (the difference between LOS and AOS), strongest signals at TCA, most rapid rate of frequency change around TCA, and maximum observed Doppler shift just at AOS and LOS. For the LUSAT-OSCAR 19 spacecraft, Fig 2 represents just such an orbit.

Our objective now is to produce a Doppler S-curve for the *next* overhead pass. With sun-synchronous satellites (and this is precisely why we chose one), the orbit tends to trace out identical ground tracks at one or two day intervals. So if we're persistent, within the next couple of days we'll see an S-curve which looks very much like Fig 2. In this example, we see the result in Fig 3.

The only thing we have to watch out for is that the two successive overhead passes must, as nearly as possible, be *identical* in relative motion. If the first observation (say, Fig 2) was made with the satellite ascending (moving from South to North), we don't want to use as our next orbit a pass in which the satellite is descending (moving from North to South). Directional beams should help to verify that both observations were made with the satellite traveling overhead in the same general direction.

Our Doppler S-curves (Figs 1 through 3) now contain all the information we require to determine orbital period, and from it, various other characteristics of the satellite and its orbit.

ESTIMATING ORBITAL PERIOD

The Doppler S-curves shown in Figs 1 through 3 depict received frequency over time, for 70-cm telemetry signals from the LO-19 satellite. Figs 1 and 2 represent two successive orbits, while the data for Figs 2 and 3 were taken one day apart. We will use the first pair of Figures to roughly estimate the orbital period of LO-19, and the second pair to refine our estimate.

Note in Fig 1 that the closest approach of the satellite to the observer (as indicated by the greatest slope of the Doppler S-curve) occurred at roughly 15 hours, 15 minutes, 36 seconds UTC. TCA for the successive orbit is noted from Fig 2 as 16:55:42, or about 100.1 minutes later. We thus have a rough estimate of orbital period, which contains an assumed error related to the Earth's rotation.

To correct the error, we note the TCAs for two successive overhead descending passes (Figs 2 and 3), which are seen to occur at 16:55:42 on one day, and then 16:26:48 the next. The elapsed time between these two overhead TCAs is thus 23:31:06 (1411.1 minutes), which must be nP, an integer multiple of the satellite's nodal orbital period.

But before we can accurately calculate P, we must have a value for the integer n. This we can determine by dividing the elapsed time between successive overhead passes, by the *estimated* orbital period. Mathematically,

$$n = \text{int}\left(\frac{nP}{P_{est}}\right) \qquad \text{[Equation 11]}$$

$$= \text{int}\left(\frac{1411.1}{100.1}\right) = 14$$

If two successive overhead TCAs are indeed separated by precisely (n = 14) orbits, then the exact orbital period must be that elapsed time divided by fourteen, or P = 100.793 minutes. Relative to this refined estimate, we see that our original estimate of orbital period, based upon two successive orbits, was off by about 0.7%. If we now compare our more exact measured value to that published for LO-19's orbital period [see Table 1], we see that we have reduced our error by roughly a factor of a hundred.

ESTIMATING OTHER ORBITAL PARAMETERS

It turns out that, for a circular sun-synchronous orbit, nearly all the important orbital parameters can be derived from the satellite's nodal period. This, after all, is why we picked this particular satellite for our experiment to begin with. I'll spare you the algebra and trig derivations; the pertinent equations are listed in the Appendix. With them, we calculate altitude, velocity, orbital increment, visibility angle, terrestrial range, access time, and Doppler shift for the LO-19 satellite.

Table 1 summarizes our results. Our "observed" values listed are either the results of direct student observation in the Penn College Telecommunications Lab, or values mathematically derived from those measured parameters. Similarly, the "theoretical" values shown are either published parameters for the LO-12 satellite given in Davidoff (1990, Appendices A and B), or values mathematically derived from those published parameters.

Note that the difference between observed and theoretical values[3] seldom exceeds a fraction of a percent. Does this mean that my students are uncannily precise? Hardly! Rather, we conclude that the experiment is structured to be forgiving of observational imprecision. We derived period, after all, by averaging elapsed time over

TABLE 1
ANALYSIS OF RESULTS

Parameter	*Units*	*Observed*	*Theoretical*	*Error*
Period	min	100.793	100.8	0.007%
Mean Altitude	km	803.9	796.4	0.9%
Velocity	m/s	7454	7458	0.1%
Increment	°W/orbit	25.2	25.3	0.4%
Max. Doppler	kHz	10	10.3	2.9%
Max. Range	km	3045	3038	0.2%
Max. LOS-AOS	min:sec	15:00	15:20	2.2%

fourteen orbits. Should this not reduce observational error by a factor of fourteen? Similarly, our Doppler S-curves depended not on *absolute* frequency measurement (always suspect), but rather on reasonably precise measurements of frequency *drift*. Even given dial calibration errors of several kilohertz (the rule rather than the exception at UHF), we can still precisely estimate the rate of frequency *change*.

CONCLUSIONS

Since their inception in 1961, Amateur Radio satellites have significantly expanded our communications horizons. They have also proved an invaluable aid in the teaching of celestial mechanics, the study of how heavenly bodies interact. If there is truly an underlying order to the Universe, it is ham satellites which may best enable our students to glimpse it. As an instructional tool, the satellite may well rank in importance second only to the chalkboard.

References

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Notes

[1]The word "beep" really is appropriate here, as the keyed signal was too long for a Morse "dit," and too short for a "dah."

[2]There is a direct analog here to successful moonbounce communication (or any other exotic DX mode, for that matter): If you know in advance the other station's call, it's about 3 dB easier to pick his call out of the noise!

[3]To motivate my students, I hesitate to call them "errors," just "differences of opinion."

Appendix

Pertinent Constants and Equations

Inertia: $F = mA$

Acceleration: $A = \frac{v^2}{r}$

Gravity: $F = \frac{GMm}{r^2}$

Velocity: $v = \left(\frac{GM}{r}\right)^{1/2}$

Period: $P = 2\,\mathrm{pi} \times \left(\frac{r^3}{GM}\right)^{1/2}$

Increment: $^{o}W = \frac{P(\mathrm{mins})}{4}$ [± Precession]

Gravitational Constant: $G = 6.673 \times 10^{-11}\ \frac{\mathrm{Nt\ m^2}}{\mathrm{kg^2}}$

Mass of the Earth: $M = 5.975 \times 10^{24}\,\mathrm{kg}$

GM Product: $GM = 3.987 \times 10^{14}\ \frac{\mathrm{m^3}}{\mathrm{s^2}}$

Mean Radius of the Earth: $R_E = 6.371 \times 10^6\ \mathrm{m}$

Doppler Shift: $f_d = \frac{f_o\, v}{c}$

Visibility Half-Angle: $\theta = \cos^{-1}\left(\frac{R_E}{R_E + h}\right)$

Max. Visibility Time: $t_{max} = \mathrm{Period} \times \left(\frac{2\,\theta}{360\ \mathrm{deg}}\right)$