TOPICS ON TRANSMISSION LINES FOR RF

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Introduction

Although the subject Transmission Lines is much presented in the literature, many topics about them seem not to be treated. Here we try to show some of their characteristics that, besides not generally well known, can help to their understanding.
We will not treat anything about Paralleled Lines or Real Stubs as other texts present them (see **References**) and some examples here are only academic with no practical uses, but still helping the global understanding.

Some important concepts

The impedance of some device or system (not necessarily electric) in the general case presents two components, the reactive (imaginary component) and the dissipative one (real component). Reactance is the system property of giving back all the energy delivered to it. As reactance is defined for a given frequency, it and the impedance are concepts rigorously valid only for well-behaved (linear and time independent) under sinusoidal regimen. It is common, however, to extend that concept for other systems and regimens, but it is only perfectly established for sinusoidal simple systems and therefore periodically repeated.
Therefore, if a system is pure dissipative, its impedance is real and, if reactive, it is pure imaginary. A general system has both components and therefore its impedance is complex.
The free space, for example, does not give back the energy of an electromagnetic wave (there is no reflection to be possible a wave return) and, so, it is dissipative.
However, a curious question arises: the line surge impedance **Zo** is real just when it is ideal, that is, not dissipative. Would that be an inconsistence? The answer is no.
If the measured impedance at the line input is **Zo**, there are two possible cases: either we have a match with a resistive load equal to **Zo** (the generator 'sees' the real load), or the line has an infinite length, when there is no reflected wave. In both cases the energy does not return and the impedance must be real (**Zo** in this case).
In a sinusoidal system with frequency **f**, we choose two dynamic variables (sinusoidal and with the same frequency **f**) **X** and **Y** such that their product is the instant power (or power density in case of waves) **P**. Therefore:

$$P = X.Y$$

However, the power is the quickness of energy transfer of the system and thus:

$$P = dE/dt$$

So we write:

$$E = \int P.dt = \int X.Y.dt$$

As **X** e **Y** are sinusoidal with frequency **f**, we may rewrite them:

$$X = Xo.\sin(2.\pi.f)$$
$$Y = Yo.\sin(2.\pi.f + \varphi)$$

where **φ** is the phase difference between both variables and **Xo** and **Yo** are their respective constant amplitudes.
Putting these definitions on the energy integral, with the integration time equal to the period **T**, we have:

$$\mathbf{E = Xo.Yo.\int_0^T sin(2.\pi.f).sin(2.\pi.f + \varphi).dt = Xo.Yo.T.cos(\varphi)/2}$$

The delivered energy is then proportional to the phase difference cosine. When **φ = 0**, **cos(φ) = 1** and therefore that energy is maximum, that is, nothing returns. If **φ = ± π/2**, **cos(φ) = 0**, the transferred energy is null, that is, everything is given back. We conclude that the system is dissipative in the first case and reactive in the second one.
So, if a circuit element is pure reactive (ideal inductor or capacitor), the voltage and the current (those variables with frequency **f** whose product is the instant power) have a phase difference of **± π/2** and, if the t element is pure dissipative (a resistance), that phase difference is zero.
Also in an ideal infinite line, as no energy return, the impedance is real and the line is 'seen' as dissipative.
In the case of a wave in the free space, the same thing happens with the electric and magnetic fields in each point of the space (the phase difference between them is zero), where here only the power (superficial) density replaces the power, but the concept is the same.
In an ideal resonant cavity, all delivered energy returns and, so, between the fields there is a phase shift of **± π/2** (or **90º**).
These results are known by the users, but without a physical analysis of the 'why', as the presented here.

Some relationships on transmission lines

The expression of the impedance Z_2 reflected at an end of an ideal transmission line (no loss) when loaded with impedance Z_1 is:

$$Z_2 = Zo.(Z_1 + j.t.Zo)/(Zo + j.t.Z_1) \quad [I]$$

where **j = √-1** is the imaginary unity, **Zo** é the line surge impedance[1] and **t = tg(2.π.l/λ)**, with **tg** the trigonometric **tangent** function, **l** the line physical length and **λ** the wavelength in the line[2].

First property (not frequently presented)

If **l** is equal to an odd multiple of **λ/8**, **t = tg[2.π.(2.n – 1)/8] = ±1** and Z_2 is:

$$Z_2 = Zo.(Z_1 + j.Zo)/(Zo + j.Z_1) \quad [II]$$

Note that, for Z_1 real and equal to R_1, the module of Z_2, $|Z_2|$, is equal to **Zo**, independently of the value of R_1:
$$|Z_2| = Zo.\sqrt{(R_1^2 + Zo^2)}/\sqrt{(Zo^2 + R_1^2)}$$

$$|Z_2| = Zo \quad [III]$$

and its phase given by $\varphi = atan[Im(Z_2)/Re(Z_2)]$, is:

$$\varphi = atan[(Zo^2 - R_1^2)/(2.Zo.R_1)] \quad [IV]$$

Therefore, if $R_1 = 0$, $\varphi = \pi/2$; if $R_1 \rightarrow \infty$, $\varphi = -\pi/2$ and if $R_1 = Zo$, $\varphi = 0$[3].

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1 **Zo is a real quantity for ideal lines and still considered real for lines with small losses.**
2 **That is, taking into account the velocity factor of the line.**
3 **Z_2 may be written as $Z_2 = Zo.\ e^{j.\varphi}$.**

This shows that, if we vary the resistive load $\mathbf{R_1}$ at the end of the line connected to a sinusoidal voltage, the resulting current will have constant module and phase variable with $\mathbf{R_1}$ from $\mathbf{-\pi/2}$ to $\mathbf{+\pi/2}$.
It would be interesting to research about some use for this property.

Second property (well known)

If $\mathbf{l = (2.n - 1).\lambda/4}$, then $\mathbf{t} \rightarrow \infty$, leading to $\mathbf{Z_2 = Zo^2/Z_1}$, that is, for a line with length equal to an odd number of one quarter wavelength, the surge impedance is the geometric average between the load and reflected impedances or $\mathbf{Zo = \sqrt{(Z_1.Z_2)}}$. Note that this expression is symmetrical between $\mathbf{Z_1}$ and $\mathbf{Z_2}$ so that we may exchange them. This means that, if we put $\mathbf{Z_2}$ as load, the reflected impedance will be $\mathbf{Z_1}$.

Third property (well known)

If $\mathbf{l = n.\lambda/2}$, then $\mathbf{t = 0}$, or $\mathbf{Z_2 = Z_1}$, that is, or a line with length equal to an integer number of half wavelength, the reflected impedance is equal to the load one, independently of **Zo**.
One can be see this with the former property, by putting two $\mathbf{\lambda/4}$ (or odd multiples of it) line pieces in series: $\mathbf{Z_1}$ transforms to $\mathbf{Z_2}$ by the first piece and $\mathbf{Z_2}$ transforms back to $\mathbf{Z_1}$ by the second piece.

Fourth property (not commonly presented)

If an ideal line of length **l** and surge impedance **Zo** is loaded with impedance $\mathbf{Z_1}$, it will reflect an impedance **Z** according to **[I]**. We ask if the same line is loaded with $\mathbf{Z_2}$, which impedance $\mathbf{Z_3}$ it will reflect, that is, which is the relationship between $\mathbf{Z_1}$ e $\mathbf{Z_3}$.
Loading the line with $\mathbf{Z_2}$:

$$Z_3 = Zo.(Z_2 + j.t.Zo)/(Zo + j.t.Z_2) \quad [V]$$

Taking $\mathbf{Z_1}$ de **[I]**, we get:

$$Z_1 = Zo.(Z_2 - j.t.Zo)/(Zo - j.t.Z_2) \quad [VI]$$

Applying **[VI]** on **[V]**, we have:

$$Z_3 = Zo.[Z_1.(1 - t^2) + 2.j.t.Zo]/[Zo.(1 - t^2) + 2.j.t.Z_1]$$

Dividing the numerator and the denominator by $\mathbf{(1 - t^2)}$:

$$Z_3 = Zo.[Z_1 + 2.j.t.Zo/(1 - t^2)]/[Zo + 2.j.t.Z_1/(1 - t^2))]$$

By observing that, if $\mathbf{t = tg(2.\pi.l/\lambda)}$, then $\mathbf{t' = 2.t/(1 - t^2) = tg[(2.\pi.(2.l)/\lambda]}$[4], we may write:

$$Z_3 = Zo.(Z_1 + j.t'.Zo)/(Zo + j.t'.Z_1) \quad [VII]$$

Therefore, $\mathbf{Z_3}$ is related to $\mathbf{Z_1}$ by the expression **[VII]**, that is, $\mathbf{Z_3}$ is the reflected impedance of a line with length **2.l** and loaded with impedance $\mathbf{Z_1}$.
This can be verified with no calculus as the done before simply remembering that $\mathbf{Z_1}$ as a load of a line with length **l** reflects $\mathbf{Z_2}$ and this, as a load of more one line piece with length **l**, reflects $\mathbf{Z_3}$. Therefore, $\mathbf{Z_3}$ is the reflected impedance of a line piece with length **2.l** when loaded with impedance $\mathbf{Z_1}$.
Let us see this graphically: as in Figure 1, the load $\mathbf{Z_1}$ reflected impedance $\mathbf{Z_2}$ on a line with length **l**; in Figure 2, the load $\mathbf{Z_2}$ reflects $\mathbf{Z_3}$ on the same line. Using the impedance $\mathbf{Z_2}$ of Figure 1 as load of Figure 2, we get the Figure 3 and, so, the expression **[VII]** holds.

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[4] **By the properties of the tangent function.**

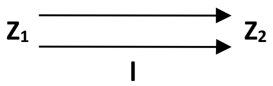

Figure 1

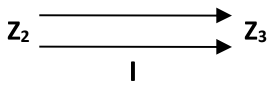

Figure 2

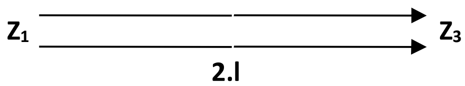

Figure 3

It is interesting to note the case where $\mathbf{Z_3 = Z_1}$, that is, when $\mathbf{Z_1}$ reflects $\mathbf{Z_2}$ and $\mathbf{Z_2}$ also reflects $\mathbf{Z_1}$.
Putting $\mathbf{Z_3 = Z_1}$ in **[VII]**, we get:

$\mathbf{Z_1.Zo + j.t'.Z_1^2 = Zo.Z_1 + j.t'.Zo^2}$ or

$\mathbf{t'.(Zo^2 - Z_1^2) = 0}$ that means:
either $\mathbf{Z_1 = Zo}$, when the line is matched to $\mathbf{Z_1}$ and $\mathbf{Z_1 = Z_2 = Z_3 = Zo}$, or $\mathbf{t' = 4.\pi.l/\lambda = 0}$, when then $\mathbf{l = n.\lambda/4}$ (**n = integer**), that is, the total length **2.l** is equal to an integer multiple of $\mathbf{\lambda/2}$, that we expected for equal reflected and load impedances.

A similar property

Suppose we have a line with length $\mathbf{l_1}$ and load $\mathbf{Z_1}$ reflecting an impedance $\mathbf{Z_2}$. We search for the minimum length $\mathbf{l_2}$ we load with $\mathbf{Z_2}$ to reflect $\mathbf{Z_1}$.
Instead of searching directly for $\mathbf{l_2}$, we search for the length $\mathbf{l_1 + l_2}$ that transforms $\mathbf{Z_1}$ into itself, that is, the first length $\mathbf{l_1}$ transforms $\mathbf{Z_1}$ into $\mathbf{Z_2}$ that is the load for the length $\mathbf{l_2}$ that transforms $\mathbf{Z_2}$ back into $\mathbf{Z_1}$, as in Figures 1', 2' and 3' bellow.

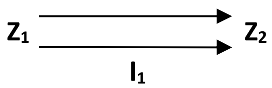

Figure 1'

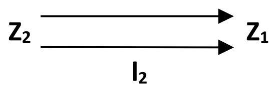

Figure 2'

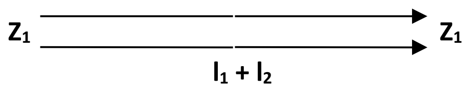

Figure 3'

Remembering that the minimum length of cable that transforms $\mathbf{Z_1}$ into itself (with any surge impedance) is $\mathbf{\lambda/2}$, then $\mathbf{l_1 + l_2 = \lambda/2}$. Therefore, the cable length that transforms $\mathbf{Z_2}$ back into $\mathbf{Z_1}$ is just $\mathbf{l_2 = \lambda/2 - l_1}$.

A curious case

Let us have a loop with total length **l** made with an ideal transmission line with surge impedance **Zo**, as in Figure 4. We ask for the impedance $\mathbf{Z_A}$ seen at the point **A** of the line[5].

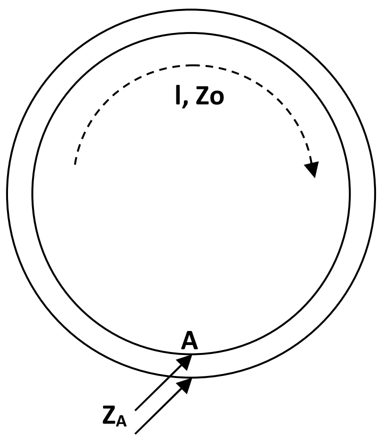

Figure 4

[5] **The point A may be any one by the line symmetry.**

We can redraw the loop as in Figure 5, where both halves[6] appear connected in parallel and with an infinite **R** load impedance. So, in Figure 5 and at its right, both conductors are insulated with no mutual connection. Figure 6 shows the details.

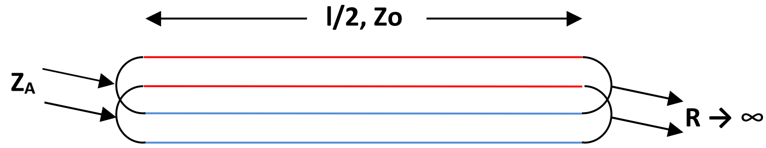

Figure 5

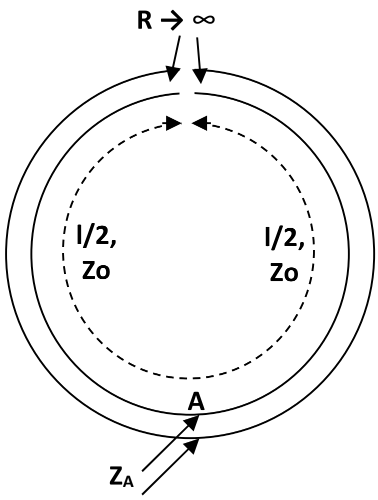

Figure 6

As both halves are in parallel, they are equivalent to a single line with the same electric length and surge impedance **Zo/2**[7]. Therefore, the impedance $\mathbf{Z_A}$ **r**eflected on **A** may be written using the expression **[I]** with **Zo** replaced by **Zo/**2, $\mathbf{Z_1}$ by **R** (that will tend to ∞) and **t** will be equal to **tg(π.l/λ)**, as **l** now is replaced by **l/2**. So, for **R** with any value:

Z_A =(Zo/2).(R + j.t.Zo/2)/(Zo/2 + j.t.R)   [VIII]

If we perform the limit when **R** → ∞, we get:

Z_A = -j.Zo/[2.tg(π.l/λ)]   [IX]

Let us see some special cases.
If the total length of the loop is **l = n.λ** (with **n = integer**), $\mathbf{Z_A}$ → ∞; this we must expect because the distance from **A** to the open load is **n.λ/2** (see 1st property).
If **l = (2.n – 1).λ/2**, $\mathbf{Z_A}$ **= 0**, because **A** sees a line with length **(2.n – 1).λ/4** open loaded (see 2nd property).

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[6] **We choose the middle point because, on it, both parts have signals with the same amplitude and phase and, so, they may be connected or disconnected without affecting anything of the circuit.**
[7] **See Reference 1.**

If **l = (2.n – 1).λ/4**, Z_A **= ± j.Zo/2** and **|**Z_A**| = Zo/2**, because **A** sees a line with length **(2.n – 1).λ/8** open loaded (see 3rd property).
We can verify that, for small **l** (compared with **λ)**, the impedance is capacitive. One could think that such a loop would result in inductive impedance[8].

Another similar case

We build a loop similar to the former one but with a 'wire exchange' on any point of the loop, as in Figure 7. We ask for the impedance Z_A**.**
The wire exchange at a line end corresponds to add an extra piece of line with length **λ/2**, as this creates a phase inversion as the wire exchange. The new line will have length **l + λ/2** and no wire exchange, falling in the former case.
Therefore, we built a line as that of Figure 4, replacing its total length **l** by **l + λ/2**. In expression **[IX]**, by making this replacement, we get:

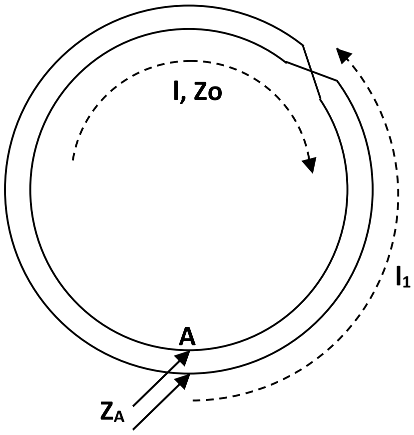

Figure 7

Z_A **= -j.Zo/[2.tg(π.l/λ + π/2)] [X]**

As summing **π/2** to the argument of a tangent function corresponds to transform it into a cotangent or the inverse of the original tangent, we have:

Z_A **= -j.Zo.tg(π.l/λ)/2 [XI]**

It is interesting to note that after inserting the line piece of length **λ/2**, the line has no more wire exchanging, that is, we lose the reference where it was initially. This shows that this initial position may be any one, leading always to the same result **[XI]**.
Let us see some special cases.

If **l = n.λ** (with **n = integer**), Z_A **= 0**
If **l = (2.n – 1).λ/2**, $Z_A \rightarrow \infty$
If **l = (2.n – 1).λ/4**, Z_A **= ± j.Zo/2** e **|**Z_A**| = Zo/2**

Minimum SWR at resonance

The module of the reflection coefficient at the load end of an ideal transmission line **Zo** is:

8 **This would be true in the case of a loop with one wire open on the point A, but not with a transmission line.**

$$|\rho| = |Z_o - Z|/|Z_o + Z| \quad [XII]$$

with **Z = R + j.X** and **Zo** a real quantity.
By the definition of the Standing Wave Ratio (SWR) **r**:

$$r = (1 + |\rho|)/(1 - |\rho|) \quad [XIII]$$

Putting **[XIII]** in **[XII]**, we have:

$$r = (|Z_o + Z| + |Z_o - Z|)/(|Z_o + Z| - |Z_o - Z|)$$ or

$$r = \{\sqrt{[(Z_o + R)^2 + X^2]} + (\sqrt{[(Z_o - R)^2 + X^2]}\}/\{\sqrt{[(Z_o + R)^2 + X^2]} - (\sqrt{[(Z_o - R)^2 + X^2]}\} \quad [XIV]$$

We want to know the smallest value of **r** when we vary **X**. This is important when we are close to the resonance of an antenna with some SWR and we vary the frequency or the antenna length. Therefore, one gets the smallest value of **r** by zeroing the derivative of **r** in respect to **X** with **R** constant in [XIV]. Rewriting **[XIV]** with $(Z_o + R)^2 + X^2 = A$ and $(Z_o - R)^2 + X^2 = B$, we get:

$$dr/dX = X.[(\sqrt{A} - \sqrt{B}).(\sqrt{B} + \sqrt{A}) - (\sqrt{A} + \sqrt{B}).(\sqrt{B} - \sqrt{A})]/[\sqrt{A}.\sqrt{B}.(\sqrt{A} - \sqrt{B})^2]$$

At the minimum, **dr/dX = 0**, which implies in **X = 0**, that is, at the resonance, the SWR is minimum for any fixed values of **Zo** and **R**. With **X = 0** in **[XIV]**, the SWR possible values are **r = Zo/R** or **r = R/Zo**, a well-known result.

References

1 - *AntenneX* Issue No. 121 – May 2007 – Note 2
2 – ARRL Antenna Book, 2007