SOLUTION METHODS FOR DIFFERENTIAL EQUATIONS

6-1. INTRODUCTION.

Application of lumped-parameter models to dynamic analysis of physical systems leads to a system description in terms of ordinary differential equations. These equations may be solved to find the system behavior by three general methods: analytical, analog computer, and digital computer. Analytical methods are essentially limited to linear equations with constant coefficients, while computer methods handle both linear and nonlinear problems. Analytical methods generally are preferred, since they allow general solutions which show the effects of system parameters directly. Both analog and digital computers can "solve" only specific numerical problems; thus the effects of parameters are revealed only by running many special cases. We will, in this chapter, show how all three approaches are used, in the case of digital methods concentrating on the popular digital simulation approach.

6-2. ANALYTICAL SOLUTION OF LINEAR, CONSTANT-COEFFICIENT EQUATIONS.

While certain nonlinear and variable-coefficient linear differential equations have closed-form analytical solutions, the majority of these equations
which arise in engineering practice have no such analytical solution and yield only to computer methods. Only for linear equations with constant coefficients do general solution techniques exist which "always work." We will here briefly review the classical operator method of solution; a condensed treatement of an alternative technique, the Laplace transform, is given in the appendix. The general form of equation which we treat is:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_1 \frac{dx}{dt} + a_0 x = f(t) \qquad \textbf{(6–1)}$$

where the *a's* are constants *and f(t)* is a known function of time. The solution proceeds in three steps:

1. Find the complementary function part of the solution, x_c.
2. Find the particular solution, x_p.
3. Add x_c to x_p to get the total solution x and apply initial conditions to evaluate the constants of integration.

A method for finding x_c which always works is available. Using the operator $D \approx d/dt$, we first write Eq. (6-1) as

$$(a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0)x = f(t) \qquad \textbf{(6–2)}$$

The *characteristic equation*

$$a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0 = 0 \qquad \textbf{(6–3)}$$

is treated as an algebraic equation in the unknown D and we must solve it for its n roots $s_1, s_2, .., s_n$. For $n > 4$, numerical approximate root-finding methods must be used and these require that the coefficients a_n to a_o be known as *numbers* rather than letters. This need to work with specific numbers is undesirable but unavoidable. Fortunately, root-finding methods can be implemented on digital computers to reduce the time and effort required at this stage of the solution. Once the roots are known, we *immediately* write down the solution x_c using a set of rules whose validity is here taken for granted but which is proven in most first courses on differential equations. For any real root s_1 which is not repeated, the solution is $C_1 e^{s_1 t}$ where C_1 is a constant of integration as yet unknown, and e is the base of natural logarithms. For a double root s_1, s_1 the solution is $C_1 e^{s_1 t} + C_2 t e^{s_1 t}$, for a triple root, $C_1 e^{s_1 t} + C_2 t e^{s_1 t} + C_3 t^2 e^{s_1 t}$ and so forth. If complex roots arise, they always come in pairs of the form $a \pm ib$ and the solution for such a pair is $Ce^{at}\sin(bt + N)$ where C and N are constants of integration. If a complex root pair is repeated $a \pm ib$, $a \pm ib$ the solution is $C_1 e^{at}\sin(bt + N_1) + C_2 t e^{at}\sin(bt + N_2)$; however, this occurs very rarely in practical problems. We thus see that once the roots are known, the solution x_c follows at once. For example, if the roots are -1, +3, +3, -2 ± i4, +3 ± i7, corresponding to a seventh order equation, the solution is:

$$x_c = C_1 e^{-t} + C_2 e^{3t} + C_4 t e^{3t} + C_5 e^{-2t} \sin(4t + \phi_1) + C_6 e^{3t} \sin(7t + \phi_2) \qquad \textbf{(6–4)}$$

While the above method for finding x_c always works, no such universal method exists for finding x_p, since it depends on the form of the forcing function *f(t)* in Eq. (6-1). No matter what method one might propose, a mathematician can always concoct a sufficiently "pathological" *f(t)* to thwart it. Thus, one must be satisfied with methods which handle a certain class of function. Fortunately, a simple method (the method of undetermined coefficients) suffices for most *f(t)*'s of engineering interest. This method will work if successive derivatives of *f(t)* ultimately become zero or repeat themselves. For example, if $f(t) = 2t^3$, all derivatives beyond the third will be identically zero and the method works. For *f(t)* == 2 sin *3t* successive derivatives give rise only to sin 3t and cos *3t* functions and the method works. If $f(t) = e^{t\wedge 2}$, successive differentiation, no matter how far it is carried, continues to produce new functional forms and the method will not work. For those cases where the method works, the particular solution x_p is written down as a sum of terms made up of every different kind of function found in *f(t)* and its derivatives, each multiplied by an undetermined coefficient. These coefficients can be found immediately by substituting x_p into the differential equation. An example illustrates the procedure.

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 4e^{-5t} \qquad \textbf{(6–5)}$$

when $t = 0^+$, $dx/dt = 0$, $x = 2.0$ (The symbol $t = 0^+$ refers to a time an infinitesimal amount after $t = 0$ and is the time instant at which "initial" conditions must be evaluated when using the classical solution method.) The characteristic equation is:

$$\boldsymbol{D^2 + 3D + 2 = 0} \qquad \textbf{(6-6)}$$

with roots $s_1 = -2$, $s_2 = -1$. The complementary function solution is:

$$x_c = C_1 e^{-2t} + C_2 e^{-t}.$$

Repeated differentiations of the forcing function $4e^{-5t}$ clearly give only terms of the form Ae^{-5t} so the method of undetermined coefficients will work. The solution $x_p = Ae^{-5t}$ is substituted into Eq. (6 -5) to give

$$25Ae^{-5t} - 15Ae^{-5t} + 2Ae^{-5t} \equiv 4e^{-5t} \qquad \textbf{(6-7)}$$

$$12Ae^{-5t} \equiv 4e^{-5t} \qquad \textbf{(6-8)}$$

$$A = 1/3 \qquad \textbf{(6-9)}$$

The complete solution is thus

$$x = x_c + x_p = C_1e^{-2t} + C_2\, e^{-t} + (1/3)e^{-5t} \qquad \textbf{(6-10)}$$

To find C_1 and C_2 we apply the initial conditions.

$$x(0) = 2 = C_1 + C_2 + (1/3) \qquad \textbf{(6-11)}$$

$$dx(0)/dt = 0 = -2C_1 - C_2 - (5/3) \qquad \textbf{(6-12)}$$

$$C_1 = -10/3 \quad C_2 = 5$$

The complete specific solution for the given initial conditions is thus

$$x = -\,(10/3)e^{-2t} + 5e^{-t} + (1/3)\, e^{-5t} \qquad \textbf{(6-13)}$$

One nice feature of the topic of differential equations is that there never is any need to accept an incorrect answer. If the solution, such as Eq. (6-13), is substituted into the original equation (6-5) and makes it an identity, and if it satisfies the initial conditions, then it *must* be the one and only correct solution.

The above simple routines will enable one to solve any ordinary linear differential equation with constant coefficients irrespective of its order (the order of the highest derivative), as long as *f(t)* can be handled by the method of undetermined coefficients. Two special cases which occur rarely, but should be mentioned, require a slightly modified procedure. If a term in x_p has the same functional form as one in x_c, the term in x_p should be multiplied by the lowest power of t which will make it different from all the x_c terms associated with the root which produced the x_c term. For example, if the right-hand side of Eq. (6-5) had been $4e^{-t}$, then x_p would have had the form Ae^{-t} the same as C_2e^{-t} in x_c. We should thus modify x_p to be Ate^{-t} before finding A. If in addition the left-hand side of Eq. (6-5) had been $D^2 + 2D + 1$, with roots $s_1 = s_2 = -1$ and $x_c = C_1e^{-t} + C_2te^{-t}$, then x_p would have to be modified to At^2e^{-t}. The second special case arises if the characteristic equation has the form

$$D^m(a_nD^{n-m} + a_{n-1}D^{n-m-1} + \ldots + a_{m+1}D + a_m) = 0, \qquad a_m \neq 0 \qquad \textbf{(6–14)}$$

When writing x_p for such a situation we must include, in addition to the usual terms, terms in the first, second,.. ., *m*th *integral* of *f(t)*.

We should at this point mention the important *principle of superposition,* which applies only to linear differential equations. If the driving function *f(t)* in Eq. (6-1) is composed of a sum of terms, $f_1(t)$, $f_2(t)$, etc., this principle allows us to find the particular solution x_p for each term of the driving function separately, and then get the total x_p by simply adding all the individual solutions. In addition to its direct mathematical utility in getting equation solutions, this principle also has two important general consequences relative to the behavior of linear systems. The first might be called the "amplitude insensitivity" of linear systems. By this we mean that if we have found the response of a system to a driving function, say, $4e^{-5t}$, if we scale this driving function up to $8e^{-5t}$ or down to $2e^{-5t}$, the response will similarly scale up or down. Nothing "new" is thus found out about the response of linear systems by changing the *size* of the driving inputs; as

long as *the form* of the driving input remains the same, the responses all are directly proportional. This follows from the superposition principle by noting that, for example, $8e^{-5t}$ can be written as $(4e^{-5t} + 4e^{-5t})$; thus the x_p for $8e^{-5t}$ is just twice that for $4e^{-5t}$. Such statements *cannot* be made for nonlinear systems; the response to an input of doubled size may be *entirely different inform* from the response to the original input. The other general consequence of superposition is that if we know how a system responds to each of two different inputs when they are separately applied, then there will be no "surprises" when they are *simultaneously* applied. That is, the behavior for the combined inputs is just the sum of the responses to the individual inputs. Again, nonlinear systems do not behave so simply; the response to a combination of inputs may show features found in *none* of the individual responses. In nonlinear systems which can go unstable, for instance, the system may be stable for each input applied separately, but unstable when they are applied together.

6-3. SIMULTANEOUS EQUATIONS.

A physical system need not be very complex for its description to require several simultaneous equations (rather than a single equation). In Fig. 6-1 a, application of Newton's law to each mass in turn leads to

$$f - K_{s1}(x_1 - x_2) - B(\dot{x}_1 - \dot{x}_2) = M_1\ddot{x}_1 \qquad \textbf{(6–15)}$$

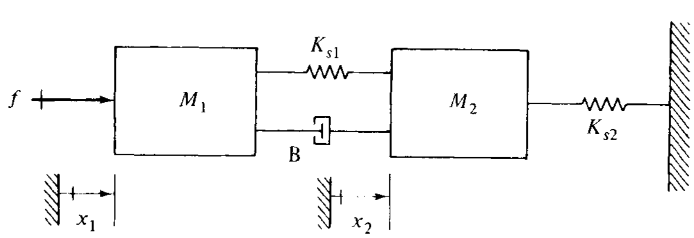
$$K_{s1}(x_1 - x_2) + B(\dot{x}_1 - \dot{x}_2) - K_{s2}x_2 = M_2\ddot{x}_2 \qquad \textbf{(6–16)}$$

Neither of these equations can be solved separately since each contains *both* of the unknowns x_1 and x_2. The pair of equations *can,* however, be solved simultaneously. Similarly, in Fig. 6-1b we get

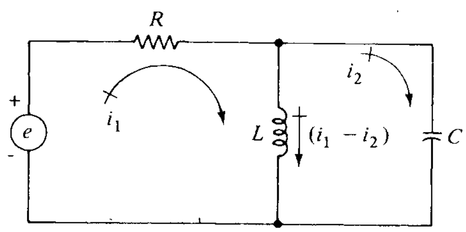
$$e - Ri_1 - L\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) = 0 \qquad \textbf{(6–17)}$$

$$-L\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) + \frac{1}{C}\int i_2\,dt = 0 \qquad \textbf{(6–18)}$$

and again the equations must be solved simultaneously. Whether the classical or Laplace transform method is used, the procedure in solving simultaneous equations basically involves reducing a set of n equations in n unknowns to a single equation in one unknown. When the classical method is used, the equations are written in operator form, whereupon they may be treated as a set of simultaneous *algebraic* equations and reduced to one equation in one unknown by any valid algebraic method, determinants being the most systematic.



(a)



(b)

FIGURE 6–1

Systems Leading to Simultaneous Equations

An example will illustrate the procedure. Suppose we have two equations in two unknowns as shown in 6-19 and 6-20.

$$\dot{x}_1 + 2x_1 - 2\dot{x}_2 + 3x_2 = 4 \qquad \textbf{(6–19)}$$

$$2\dot{x}_1 + x_1 + \dot{x}_2 - x_2 = 2t \qquad \textbf{(6–20)}$$

When $t = 0^+$, $x_1 = 1$, $x_2 = -2$.

In operator form these become

$$(D + 2)x_1 + (-2D + 3)x_2 = 4 \qquad (6\text{-}21)$$

$$(2D + 1)x_1 + (D - 1)x_2 = 2t \qquad (6\text{-}22)$$

We now treat these as algebraic equations in the unknowns x_1 and x_2. with the D operator carried along as if it were an ordinary parameter. We wish to reduce the set of equations to a single equation in x_1 and another single equation in x_2. Using determinants as in algebra we get

$$= \frac{\begin{vmatrix} 4 & -2D + 3 \\ 2t & D - 1 \end{vmatrix}}{\begin{vmatrix} D + 2 & -2D + 3 \\ 2D + 1 & D - 1 \end{vmatrix}} = \frac{(D - 1)4 - (-2D + 3)2t}{(D + 2)(D - 1) - (2D + 1)(-2D + 3)}$$

$$= \frac{-6t}{5D^2 - 3D - 5} \qquad \textbf{(6–23)}$$

Note that in the numerator the (D-1) term *operates* on the constant 4 giving -4 and -(*-2D*+3) *operates* on 2t giving (+4-6t). Cross-multiplying Eq. (6 23) gives

$$(5D^2 - 3D - 5)x_1 = -6t \qquad \textbf{(6-24)}$$

the desired single equation in x_1. Similarly, for x_2

$$x_2 = \frac{\begin{vmatrix} D+2 & 4 \\ 2D+1 & 2t \end{vmatrix}}{\begin{vmatrix} D+2 & -2D+3 \\ 2D+1 & D-1 \end{vmatrix}} = \frac{(D+2)2t - (2D+1)4}{5D^2 - 3D \cancel{+} 5}$$

$$= \frac{4t-2}{5D^2 - 3D \cancel{+} 5} \qquad \textbf{(6–25)}$$

$$(5D^2 - 3D \cancel{+} 5)x_2 = 4t - 2 \qquad \textbf{(6–26)}$$

Note that the characteristic equation $5D^2$ - $3D$ -5 = 0 is the same no matter which unknown, x, or x,, is being solved for. We can see that this will be true for the general case of n equations in n unknowns since the denominator determinant is the same no matter which unknown is being considered. Since the physical system is described by the whole set of equations, this means a linear system, no matter how complex, has only one characteristic equation. To complete the solution we find the characteristic equation roots to be 1.34 and –0.74 and thus

$$x_{1c} = C_1 e^{1.34t} + C_2 e^{-0.74t} \qquad \textbf{(6–27)}$$

$$x_{2c} = C_3 e^{1.34t} + C_4 e^{-0.74t} \qquad \textbf{(6–28)}$$

$$x_1 = -6t + 18 + C_1 e^{1.34t} + C_2 e^{-0.74t} \qquad \textbf{(6–29)}$$

$$x_2 = 4t + 10 + C_3 e^{1.34t} + C_4 e^{-0.74t} \qquad \textbf{(6–30)}$$

x_{1p}= -(6/5)t-(18/25) and x_{2p}=(4/5)t+(22/25)

It appears we have four constants of integration to be found and only two initial conditions; however, the four constants of integration are not really all independent, so the problem is solvable. Perhaps the easiest way to find the needed constants is to generate additional initial conditions from those given and the system equations. That is, the basic equations (6-19) and (6-20) are true at every instant of time, including t = 0, and can be used to find the needed initial conditions. At t = 0, (6-19) and (6-20) give

$$\dot{x}_1 + 2 - 2\dot{x}_2 - 6 = 4 \qquad \textbf{(6–31)}$$

$$2\dot{x}_1 + 1 + \dot{x}_2 + 2 = 0 \qquad \textbf{(6–32)}$$

and these yield dx_2(0)/dt=-3.8 and dx_1(0)/dt=0.4. We now have sufficient initial conditions to solve for C_1, C_2, C_3, and C_4 and get the complete specific solutions for x_1 and x_2.