Distributed Amplification*

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Summary—**This paper presents a new principle in wide-band amplifier design. It is shown that, by an appropriate distribution of ordinary electron tubes along artificial transmission lines, it is possible to obtain amplification over much greater bandwidths than would be possible with ordinary circuits. The ordinary concept of "maximum bandwidth-gain product" does not apply to this distributed amplifier. The high-frequency limit of the distributed amplifier appears to be determined by the grid-loading effects.**

The distributed amplifier provides means for designing amplifiers either of the low-pass or band-pass types. The low-pass amplifiers can be made to have a uniform frequency response from dc to frequencies as high as several hundred Mc using commercially available tubes.

The general design considerations included in this paper are: The effect of improper termination of transmission lines; methods for controlling the frequency response and phase characteristic; the design which provides the required gain with fewest possible number of tubes; and a discussion of high-frequency limitations. The noise factor of the amplifier is evaluated.

Practical amplifiers, designed according to the principles described in this paper, have been built and have verified the theoretical predictions. Experimental work will be described in a forthcoming paper.

I. Introduction

With the expansion of the electronic art, there has been a steadily increasing demand for still wider-bandwidth amplifiers. The conventional techniques of cascading amplifier stages have been explored thoroughly in the recent years, and it has been shown[1–3] that there is a maximum "bandwidth-gain product" for a given tube type, no matter how complex is the coupling system between stages. Aside from the practical difficulties of attaining this maximum, this basic limitation determines the maximum bandwidth that can be obtained with conventional tubes and circuits.

The introduction of the traveling-wave concepts[4,5] has provided a new technique for wide-band amplification at microwave frequencies. In principle, it is possible to build traveling-wave tubes which will amplify low frequencies as well as microwaves; on the other hand, the traveling-wave tube must be electrically long, and practical limitations make it improbable that such tubes will be available for frequencies much below 1000 Mc. To date, no practical solution for extremely broadband "video" amplifiers has been found.

The distributed amplifier to be described below provides means for designing amplifiers which have flat frequency response from low audio frequencies (and dc if necessary) to frequencies as high as several hundred Mc. This is accomplished by applying traveling-wave concepts to the "video" frequency region. By this method, as will be shown, the conventional restrictions on bandwidth are completely removed, the high-frequency limit being determined entirely by high-frequency effects within the tube proper, and not by the circuit effects outside of the tubes.

It should be pointed out that the basic idea described in this paper is not new, being first disclosed by Percival.[6] However, for reasons which are not clear to the authors, there does not seem to be further discussion of this idea in the literature. The name "distributed amplifier" is due to the authors of this paper.

II. Basic Principles

It has been shown by Wheeler[1] and others that the frequency limit of a conventional video amplifier is determined by a factor which is proportional to the ratio of the transconductance G_m of the tube to the square root of the product of the input and output capacitances. Clearly, it does not help matters simply to parallel tubes; the resulting increase in G_m is compensated for by the corresponding increase in the combined capacitances. The distributed amplifier about to be described overcomes this difficulty by paralleling the tubes in a special way, in which the capacitances of the tubes may be separated while the G_m of the tubes may be added almost without limit and not affect the input or output impedance of the device. In its simplest form, this result is achieved by using the tube capacitances as the shunting elements in an artificial transmission line.

Fig. 1 shows the structure of the distributed amplifier.

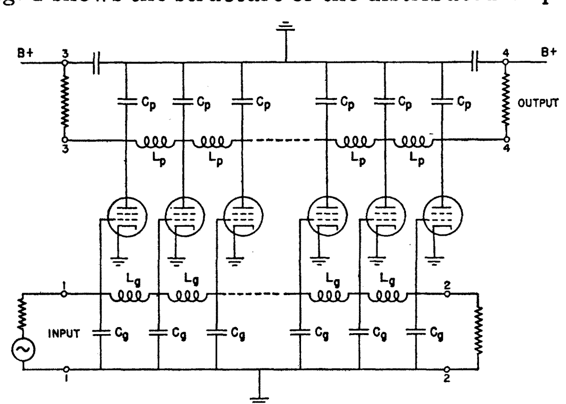

Fig. 1—Basic distributed amplifier.

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[1] H. A. Wheeler, "Wide-band amplifiers for television," Proc. I.R.E., vol. 27, pp. 429–438; July, 1939.

[2] W. W. Hansen, "Maximum gain-bandwidth product in amplifiers," *Jour. Appl. Phys.*, vol. 16, pp. 528–534; September, 1945.

[3] Hendrick W. Bode, "Network Analysis and Feedback Amplifier Design," D. Van Nostrand Co., Inc., New York, N. Y., 1945, chap. XVII.

[4] J. R. Pierce and L. M. Field, "Traveling-wave tubes," Proc. I.R.E., vol. 35, pp. 108–111; February, 1947.

[5] R. Kompfner, "The traveling-wave tube as amplifier at microwaves," Proc. I.R.E., vol. 35, pp. 124–128; February, 1947.

[6] W. S. Percival, British Patent Specification No. 460,562, applied for, July 24, 1936.

Between the input terminals *1-1* and terminals *2-2* there is an artificial transmission line, which consists of the grid-cathode capacitance of the tubes C_g, and the inductance between tubes (or sections) L_g. Then the characteristic impedance of the grid line is

$$Z_0 = \sqrt{\frac{L_g}{C_g}}. \tag{1}$$

If the proper terminating impedance is connected to terminals *2-2*, and if this transmission line is assumed to be dissipationless, then it can be shown that the driving-point impedance at terminals *1-1* is *independent of the number of tubes so connected.* In a like fashion, a second transmission line is formed by making use of the plate-to-cathode capacitances to shunt another set of coils L_p. The impedance of the plate line is similarly independent of the number of tubes (sections). Impedances connected to terminals *3-3* and *4-4* are intended to be equal to the characteristic impedance of the plate line. The impedance connected to terminals *2-2* will be called the *grid termination;* that connected to terminals *3-3* will be called the *reverse termination;* and the impedance connected to terminals *4-4* will be called the *plate termination.* Terminals *4-4* are the *output terminals.*

The two transmission lines so formed are made (by design) to have identical velocities of propagation.

A generator connected to the *input terminals 1-1* will cause a wave to travel along the grid line. As this wave arrives at the grids of the distributed tubes, currents will flow in the plate circuits of the tubes. Each tube will then send waves in the plate line in both directions. If the reverse termination is perfect, the waves which travel to the left in the plate line will be completely absorbed, and will not contribute to the output signal. The waves which travel to the right in the plate line all add *in phase*, as can be verified by examining the various possible paths between the input and output terminals. Thus, the output voltage is directly proportional to the number of tubes. The net result is that the effective G_m of this distributed "stage" may be increased to any desired limit. Thus, no matter how low the gain of each tube (section) is (even if it is less than unity), as long as the gain per section is greater than the transmission-line loss of the section, the signal in the plate line will increase and can be made to be as large as one desires by merely using a sufficient number of tubes.

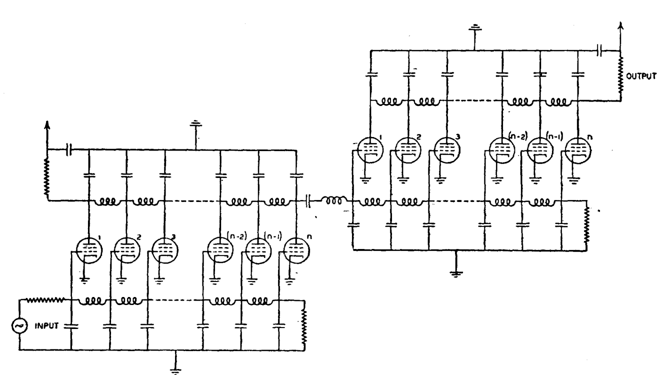


Fig. 2—Two-stage distributed amplifier, having n tubes per stage.

When sufficient gain has been accumulated in one distributed-amplifier stage, then such stages can be cascaded in the normal manner as shown in Fig. 2.

III. Cascading of Stages

It can be easily shown that there is an optimum method of dividing the tubes into groups. Appendix I shows that the least number of tubes that is required to produce a desired total gain G results when each stage[7] has a gain of ϵ (the Naperian logarithmic base, equal to 2.72). Each such stage has n sections, and the stages are cascaded m times. Thus, there are mn tubes in such an amplifier.

If a total gain G is required, then the number of cascaded stages that should be used is m (see Appendix I).

$$m = \log_\epsilon G. \tag{2}$$

The total number of sections that must be used in each stage must be large enough to provide a gain of ϵ for the stage. The number n obviously depends upon the bandwidth desired and upon the type of tube to be used. It is convenient to express the high-frequency figure of merit of a tube as a *bandwidth index frequency*[1]; i.e., the maximum bandwidth over which unity gain may be obtained. The number of sections in each stage will then be a simple function of the ratio of the desired bandwidth to this index frequency. It is

$$n = 2\frac{f_c}{f_0}\epsilon \tag{3}$$

where

f_c = high-frequency cutoff of the amplifier

f_0 = Wheeler's bandwidth-index frequency $= \dfrac{G_m}{\pi\sqrt{C_gC_p}}$

$\epsilon = 2.72.$

The number of sections required to produce a gain of ϵ is plotted in Fig. 3 for the case under discussion, and also for the conventional cascade amplifier. It is evident from this figure that the distributed amplifier is the *only* means available for amplification when the maximum frequency desired is greater than the bandwidth index frequency of the tube being used. Further, it is usually found that it is impractical to achieve much more than 50 per cent of the theoretically available bandwidth with conventional circuits; this is so because the theoretical limit requires the use of extremely complex coupling circuits, which can hardly be considered practical and

[7] The following nomenclature will be used in this paper: Each electron tube with its section of transmission line will be called a *section;* the gain of the section will be called A_0. n such sections form a *stage*, with a gain A. When such stages are cascaded in the conventional manner, they are called *cascaded stages*, with a gain G.

which increase the stray capacitance to ground. This is not the case in the distributed amplifier.

The basic ideas presented in the above discussion were in terms of the low-pass filter structure. It is obvious that the principle is equally applicable to band-pass filters. The distributed amplifier can be made to operate even in cascaded form at frequencies down to dc by utilizing well-known dc amplifier techniques.[8]

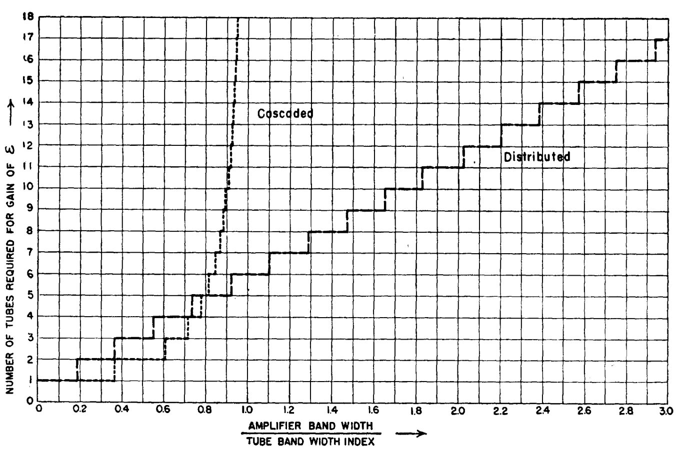


Fig. 3—Number of tubes required to produce a gain of ϵ in cascaded and in distributed amplifiers.

IV. Frequency-Response Characteristics

The following discussion of the frequency-response characteristics of the distributed amplifier will be carried out in terms of the low-pass structure of the type shown in Figs. 1 and 2. Several of the equations below are of a general type, however, and only simple modifications need to be made to make the analysis applicable to other possible structures.

The voltage gain of the amplifier consisting of n sections per stage and m cascaded stages is

$$G = \left[\frac{nG_m}{2}\sqrt{Z_{01}Z_{02}}\right]^m \tag{4}$$

where the symbols are, as before,

G = total gain

Z_{02} = characteristic impedance of the plate line

Z_{01} = characteristic impedance of the grid line.

For the case shown in Figs. 1 and 2, and assuming that the two transmission lines are identical,

$$Z_{01} = Z_{02} = \frac{R}{\sqrt{1 - x_k^2}} \tag{5}$$

where

$$x_k = \frac{f}{f_c}$$

$$R = \frac{1}{\pi f_c C}$$

f = frequency

f_c = cutoff frequency of the transmission lines.

Under these conditions, the gain of the distributed amplifier becomes

$$G = \left[\frac{nG_m}{2}\cdot R\right]^m (1 - x_k^2)^{-m/2}. \tag{6}$$

The second factor of this equation shows that the gain of the simple structures shown in Figs. 1 and 2 will be a function of frequency. This is due to the fact that the mid-shunt characteristic impedance of a constant-K filter section (these lines obviously are constant-K sections) rises rapidly as the cutoff frequency is approached. This, in turn, causes the gain of the amplifier to increase sharply near cutoff, producing a large undesired peak. In principle, this peak can be equalized, but this becomes increasingly difficult as the number of cascade stages increases.

There are cases where the peak at the high-frequency end is not harmful and may be even beneficial. However, there are several methods which can be used to eliminate this peak. Three of these methods are discussed below.

(a) *Paired-Plate or Paired-Grid Connection*

Fig. 4 (a) shows a somewhat different arrangement of electron tubes along the transmission lines from those previously discussed. The grids of the tubes are still

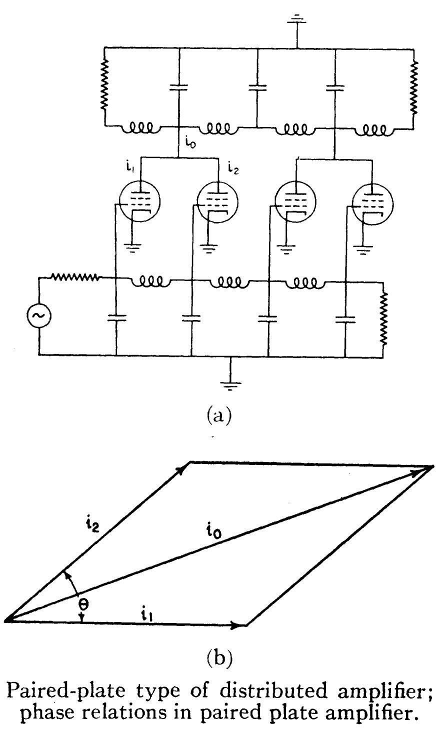


Fig. 4—(a) Paired-plate type of distributed amplifier; (b) current phase relations in paired plate amplifier.

connected periodically along the grid line, but the plates are paired as shown, with a dummy capacitance being placed at the point where the plate capacitance is now missing. This particular arrangement of tubes will be called the *paired-plate* connection. It is possible to pair grids and leave the plates arranged periodically. This is

[8] E. L. Ginzton, "D-c amplifier design technique," *Electronics*, vol. 17, pp. 98–102; March, 1944.

called the *paired-grid* connection. The action of the two circuits is similar, and only the paired-plate connection will be discussed below.

The operation of this paired-plate circuit can be understood by referring to the vector diagram of the plate currents at the common junction, shown in Fig. 4 (b). Let i_1 be the current in one of the tubes, and i_2 the current in the other. The phase angle between i_1 and i_2 is determined by the phase shift between the grids of the two tubes[9] and is given by

$$\theta = 2 \sin^{-1} x_k \tag{7}$$

where x_k is the normalized frequency of the section, as defined above. The resultant current vector is a function of x_k, and is

$$i_0 = 2\sqrt{1 - x_k^2}. \tag{8}$$

It is evident that this factor is the reciprocal of the characteristic-impedance function of the section. Thus, the voltage developed in the plate line, being the product of i_0 and Z_{02}, will be constant over the pass band of the filter.

By leaving some of the plates unpaired, the gain of a stage can be made to have a frequency response which is intermediate between the flat characteristic of the completely paired stage and the rising characteristics of the constant-K sections. The control of the degree of rise in gain is a very valuable feature of this circuit. This increase in gain can be used to compensate for the decrease in gain which is due to attenuation in the transmission line at high frequencies.

Since the plate-to-cathode capacitance of most pentodes is about one-half of the grid-to-cathode capacitance, the addition of extra capacitance in the plate line does not reduce materially the design cutoff frequency.

(b) *Negative Mutual-Inductance Circuit*

The method of improving the frequency response about to be described is slightly more complicated from the original design viewpoint, but has several desirable features which are believed to be of great importance. The basic connection is shown in Fig. 5 (a), which differs from Fig. 1 only in that the adjacent coils are wound on

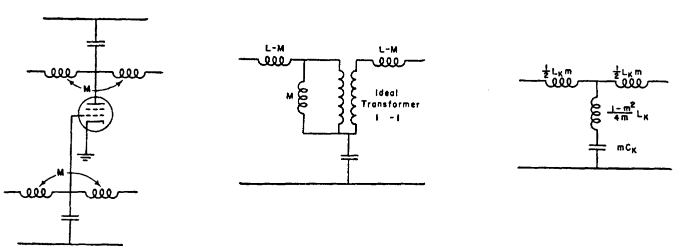


Fig. 5—(a) Circuit using mutual coupling between coils; (b) circuit equivalent to (a) by transformer theory; (c) m-derived filter circuit equivalent to (a) and (b).

[9] E. A. Guillemin, "Communication Networks," vol. II, McGraw-Hill Book Co., Inc., New York, N. Y., 1935, p. 316.

the same form and in the same direction, and have a large coefficient of coupling. Each section can be resolved by conventional transformer theory into Fig. 5 (b). By proper design, this can be equated to the usual m-derived section shown in Fig. 5 (c). If the mutual inductance is negative, as it is in the case being discussed, the constant m will be greater than unity. This has two very desirable features. First, m greater than unity leads to a more linear phase shift through the stage. This becomes particularly important if a large number of stages are to be cascaded. Secondly, $m>1$ leads to a larger value of capacitance C than would be called for if constant-K sections were to be used instead. For a given capacitance C, then, it is possible either to increase the gain per section for the same bandwidth, or to increase the bandwidth for the same gain.

Equation (9) shows the grid-to-plate gain, and (10) the phase shift for a stage with n tubes connected as shown in Fig. 6. This equation is derived in Appendix II.

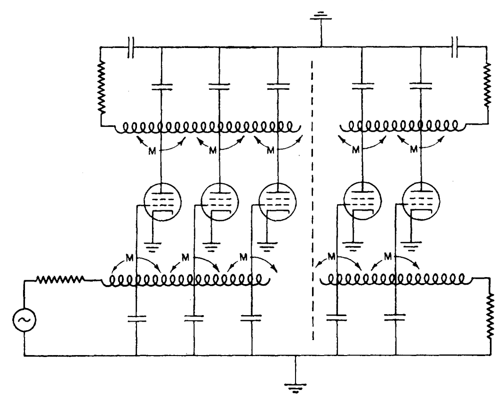


Fig. 6—An n-stage distributed amplifier using mutual coupling between coils.

$$G = \frac{R_0 G_m}{2} \cdot \frac{nm^3}{[m^2 - (1 - m^2)x_k^2]\sqrt{m^2 - x_k^2}]} \tag{9}$$

$$\phi = 2n \tan^{-1} \frac{m x_k}{\sqrt{m^2 - x_k^2}} \tag{10}$$

where

$$R_0 = \frac{K}{\pi f_m C_g} \tag{11}$$

$$L_k = \frac{C_g R_0^2}{m} \tag{12}$$

$$L = \frac{1 + m^2}{4m^2} L_k, \qquad M = \frac{1 - m^2}{4m^2} L_k \tag{13}$$

in which

f_m = maximum frequency required with amplitude or phase tolerance ϵ

K = coverage factor, to be determined from Fig. 9 for a desired value of tolerance ϵ

$x_k = f/f_0 = (f/f_m)K$ normalized frequency function

m = design parameter selected from Fig. 9 for desired ϵ.

The time delay through the stage is the derivative of the phase shift with respect to angular frequency, and is

$$\tau = \frac{nd\phi}{d\omega} = \frac{nd\phi}{dx_k}\cdot\frac{1}{\omega_0}$$
$$= \frac{nm^3}{[m^2 - (1 - m^2)x_k^2][m^2 - x_k^2]^{1/2}\pi f_0} \text{ seconds.} \quad (14)$$

It is interesting to observe that both the gain and delay functions are the same except for numerical constants. Figs. 7 and 8 show the relative gain, time delay, and phase shift as a function of normalized frequency x_k for four values of m.

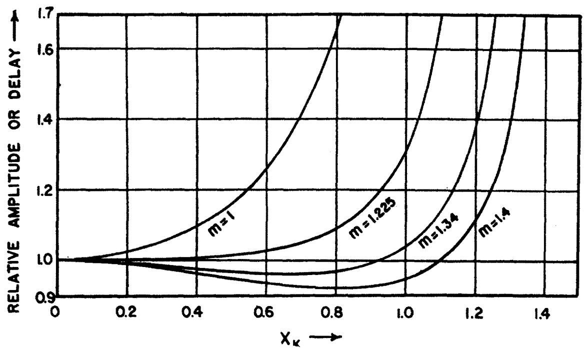


Fig. 7—Relative gain and time delay of amplifier with mutual coupling versus normalized frequency.

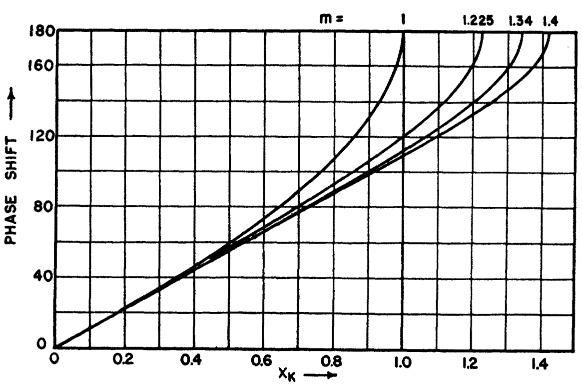


Fig. 8—Phase shift of amplifier using mutual coupling versus normalized frequency.

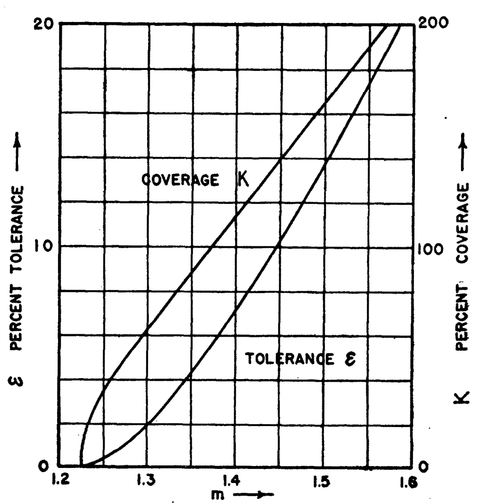


Fig. 9—Per cent tolerance or phase linearity and per cent of band covered for amplifier with mutual coupling.

Fig. 9 is designed to permit the selection of any desired tolerance in either phase or amplitude linearity as a function of per cent band coverage K over which tolerance may be maintained.

(c) *The Bridged-Tee Connection*

The third method of equalizing the frequency response is by means of the bridged-tee connection shown in Fig. 10 (a). By simple transformer theory this is equivalent to the circuits shown in Figs. 10 (b) and 10 (c). Fig. 10 (c) corresponds to a line having mutual coupling between coils and shunted by an impedance Z_c. If Z_c is the capacitance C_1 and Z_d is C_g, the tube capacitance, then, using Fig. 10 (d), the circuit may be

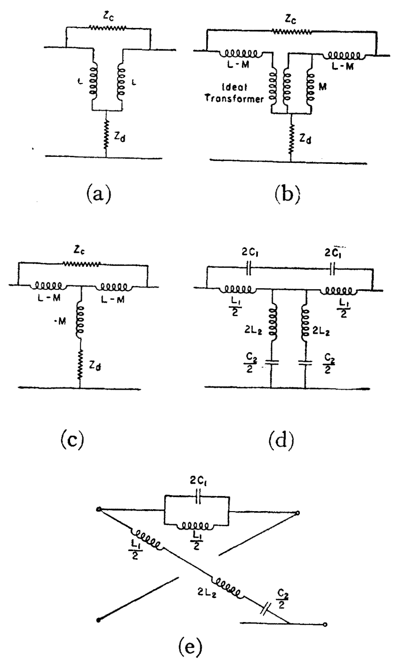


Fig. 10—Equivalent circuits for bridged-tee connection.

converted into a lattice (Appendix III) having the arms

$$Z_a = \frac{\frac{1}{4}\frac{L_1}{C_1}}{\frac{1}{2}j\omega L_1 + \frac{1}{2j\omega C_1}}$$
$$Z_b = \frac{1}{2}j\omega L_1(1+\alpha) + \frac{2}{j\omega C_g} \quad (15)$$

where

$$\alpha = 4\frac{L_2}{L_1}.$$

This lattice is shown in Fig. 10 (e). Its characteristic impedance is

$$Z_0 = \sqrt{\frac{L_1}{C_g}\,\frac{1 - \omega^2\frac{L_1C_g}{4}(1+\alpha)}{1-\omega^2 L_1C_1}}. \quad (16)$$

If $C_g(1+\alpha)/4=C_1$, this equation is independent of frequency and the impedance becomes

$$Z_0 = \sqrt{\frac{L_1}{C_g}} = R_0. \tag{17}$$

If x_k is defined as before, the phase shift θ and the time delay τ per section become

$$\theta = 2 \tan^{-1} \frac{x_k}{1 - x_k^2(1+\alpha)} \tag{18}$$

$$\tau = \frac{\partial\theta}{\partial\omega} = \frac{1}{\pi f_p} \frac{1 + x_k^2(1+\alpha)}{[1 - x_k^2(1+\alpha)]^2 + x_k^2} \text{ seconds.} \tag{19}$$

The stage gain A for n sections will be

$$A = n \frac{R_0 G_m}{2} \frac{1}{[1 - x^2(1+\alpha)]^2 + x^2}$$

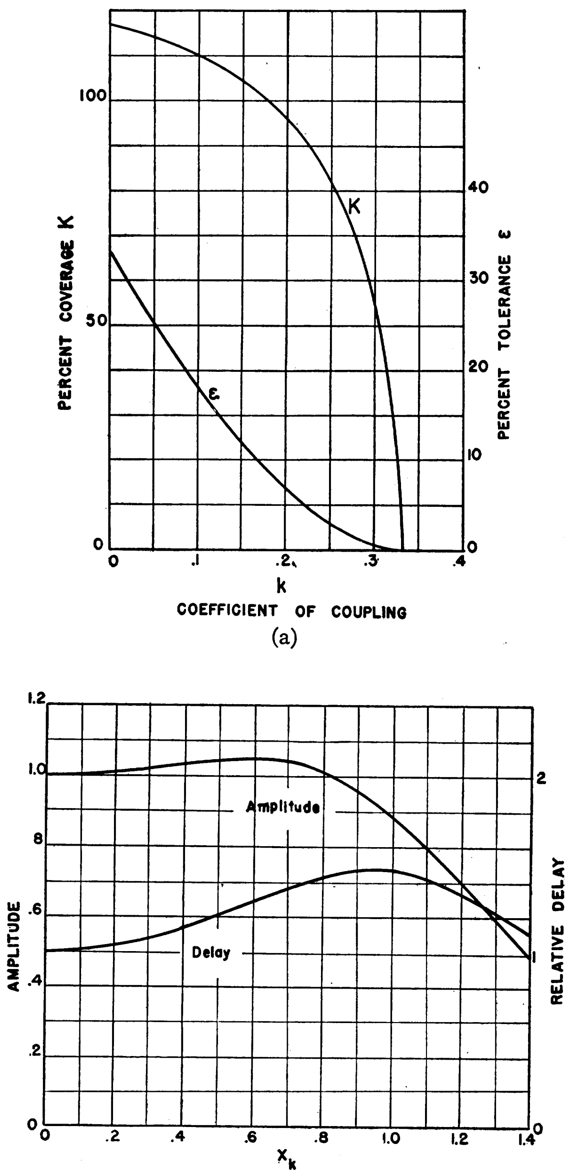


Fig. 11—(a) Coverage and tolerance for bridged-tee amplifier; (b) gain and time delay for bridged-tee amplifier.

where

$$R_0 = \frac{1}{\pi C_g f_m} K. \tag{20}$$

The parameters L and C_1 are given by

$$L = 1/2C_g R_0^2 \frac{1}{1-k} \tag{21}$$

$$C_1 = 1/4\left(\frac{1-k}{1+k}\right)C_g \tag{22}$$

in which k, f_m, x_k are defined previously, but being selected in this case from Fig. 11 (a), which gives ϵ and K as a function of coefficient of coupling k.

The gain and time delay for a typical case of $k=0.215$ are shown in Fig. 11 (b).

V. The Effect of Improper Terminations of Lines

In all of the discussion above, it was assumed that the lines were perfectly terminated. In the first place, it should be pointed out that, in general, the artificial lines need to be terminated by proper half-sections and resistors equal to the characteristic impedance of the lines. This is done in a conventional way, and will not be described in detail. In any practical situation, however, the terminations cannot be perfect; there can be reflections from all four sets of terminals. The effect of these reflections may be understood by reference to Fig. 12 (a), which is a schematic diagram of one stage of a distributed amplifier. It shall be assumed that the

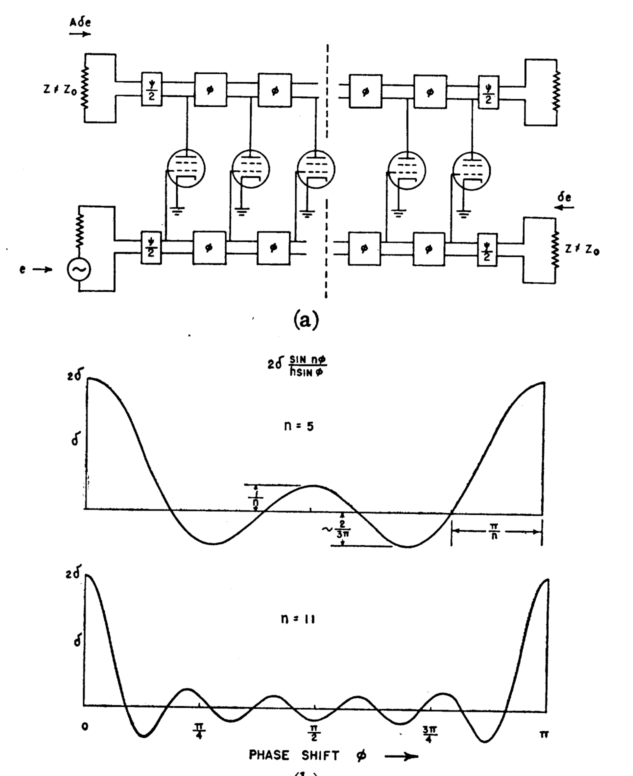


Fig. 12—(a) Diagram of distributed amplifier showing phase shift and reflection from terminations; (b) ratio of signal to reflected voltages.

lines are dissipationless, and that all sections are identical. Each stage has a phase shift of ϕ degrees, and each end of each line is terminated by a terminal half-section. The terminal half-sections will be assumed to have a phase shift of $1/2\psi$ degrees. If a signal e is introduced into the grid line, then a portion of that signal will be reflected from the grid termination. If δ is the reflection coefficient, then the reflected wave will have an amplitude δe, where $\delta = (Z_L - Z_0)/(Z_L + Z_0)$. For the sake of simplicity, it will be assumed that secondary reflections from the input and from the plate termination are negligible. The reflected voltage δe will appear at the grids of the various tubes and will add vectorially to the original wave. In a similar fashion, reflections may be expected from the reverse termination in the plate line. The net voltage at the output of the distributed amplifier is then a vector sum of all of these voltages. The net voltage due to reflections alone is

$$E_r = 2A_0 e\delta\big[\epsilon^{j[2\psi+(n-1)\phi]} + \epsilon^{j[2\psi+(n+1)\phi]} + \cdots + \epsilon^{j[2\psi+(2k+n-1)\phi]} + \cdots + \epsilon^{j[2\psi+3(n-1)\phi]}\big]. \tag{23}$$

The voltage due to the signal at the output terminals, neglecting reflections, is

$$E_s = A_0 e n \epsilon^{j[\psi+(n-1)\phi]} \tag{24}$$

where

A_0 = amplification per section
e = input signal
n = number of tubes per stage.

The ratio of the reflected voltage to the signal voltage is then given by (E_r/E_s), and it is

$$\frac{E_r}{E_s} = \frac{2\delta}{n}\sum_{k=0}^{k=n-1}\epsilon^{j[\psi+2k\phi]} \tag{25}$$

$$= 2\delta\frac{\sin n\phi}{n\sin\phi}\epsilon^{j[\psi+(n-1)\phi]}. \tag{26}$$

This equation predicts the importance of the reflections. The magnitude of this function is plotted in Fig. 12(b) for $n=5$ and $n=11$. It is evident from (26) and from Fig. 12(b) that the relative magnitude of the reflected voltages near the center of the band depends upon $(2\delta/n)$ and that the larger peaks are displaced toward the edges of the band.

From a practical standpoint, the reflection factor for low values of ϕ, i.e., at the lower frequencies, may be made nearly zero. The larger peaks as ϕ approaches π tend to move toward the edges of the useful range of the amplifier. Furthermore, the concave phase characteristic of the normal constant-K section will still further crowd these larger peaks toward the upper end of the frequency band. It is evident, then, that as the number of sections n is increased, the seriousness of small mismatches is reduced.

The actual output voltage is the vector sum of the nominal output signal and the reflected signal. Fig. 13 shows the magnitude of the variations in the output voltage for $n=5$ and $n=11$ when it is assumed that δ is

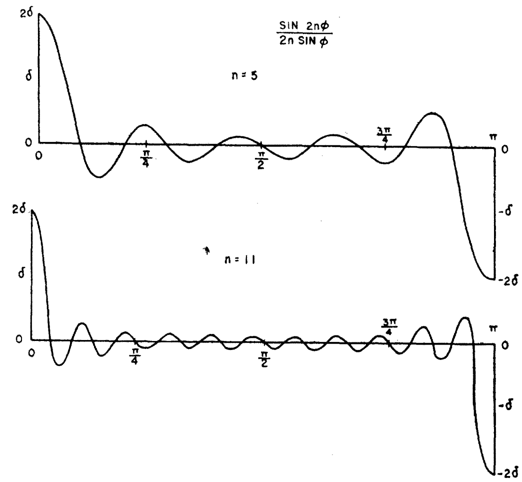


Fig. 13—Variations in output voltage.

small compared to 1 and $\phi=\psi$. If ψ is less than ϕ, as is usually the case, the effect on the curve as shown in Fig. 13 will be to crowd it toward the right and displace it slightly downward in accordance with (27).

$$E_{\text{out}} \sim E_s\left[1 + 2\delta\left(\frac{\sin(2n\phi+\psi-\phi)}{2n\sin\phi} - \frac{\sin(\psi-\phi)}{2n\sin\phi}\right)\right]. \tag{27}$$

When the number of sections is small, i.e., less than four, the value of m in the terminal half-section may be so selected that the characteristic impedance and the terminating resistance will be equal $(\delta=0)$ at a frequency coinciding with one of the maxima of Fig. 12(b). This will further tend to reduce the reflection effect from an imperfect termination.

VI. Tapered Plate Lines

In cases where it is desired to operate a distributed amplifier into a lower impedance than the optimum design impedance of the plate line, the so-called tapered line sections in the plate circuit may be used. Referring to Fig. 14, the first tube operates into a section of line

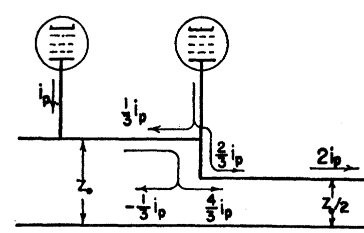


Fig. 14—Current distribution in tapered line.

with a characteristic impedance of Z_0 which is unterminated, and all the plate current i_p will flow down this section. If the next section has a characteristic impedance $1/2\ Z_0$, there will be a reflected current from this discontinuity of $-1/3\ i_p$ and, in accordance with Kirchhoff's law, a current will flow into the new section of $4/3\ i_p$. However, the current flowing into this junction from the second tube will produce a current of $1/3\ i_p$ back down the line exactly cancelling the reflected current, and a forward current of $2/3\ i_p$ which, added to the forward current of the first tube, will give $2\ i_p$ flowing in the new section. At the next junction, the third section should have a characteristic impedance equal to $2/3$ of the preceding section, or $1/3\ Z_0$. It is evident, then, that the output impedance of the line will be Z_0/n, where Z_0 is the initial impedance and n is the number of sections per stage. The entire current of the output tubes may thus be effectively used in the load without the necessity of half the current flowing in the load and half the current flowing in the reverse termination.

VII. High-Frequency Effects

When an attempt is made to build an amplifier embodying the principles of the distributed amplifier and operating at frequencies above 100 Mc, the effect of lead inductance, grid loading, and line loss must be taken into account.

(a) Incidental Dissipation

It is well known that series resistance and shunt conductance will produce attenuation in a filter. Equation (28) is a good approximation of the effect of such dissipation. It is to be noted from this equation,[10]

$$\alpha = \frac{x_k}{2}\left(\frac{1}{Q_c} + \frac{1}{Q_L}\right)\frac{d\phi}{dx_k} \tag{28(a)}$$

$$= \left(\frac{G}{2c} + \frac{R}{2L}\right)\frac{d\phi}{d\omega} \tag{28(b)}$$

where

α = attenuation in nepers
Q_c = the Q of the capacitors
Q_L = the Q of the coils
x_k = the normalized frequency function
G = the shunt conductance across the capacitance C
R = the resistance in series with inductance L
ϕ = the phase shift of the section in radians,

that dissipation produces an attenuation in the pass band proportional to the sum of the reciprocals of the Q's of the coils and capacitors and proportional to the normalized slope of the phase function times the normalized frequency function x_k. As the phase function of a constant-K section is concave and rises sharply near cutoff, a marked increase in attenuation will occur near the cutoff frequency. The advantage of a linear phase function such as that obtained from sections utilizing negative mutual impedance is also immediately evident when considering the effects of incidental dissipation.

[10] See p. 447 of footnote reference 9.

(b) Lead Inductance

Lead inductance in the grid and plate circuits has the effect of reducing the cutoff frequency and producing a peak near cutoff. The use of negative mutual inductance can completely compensate for this effect. The constants L and M of the negative-mutual-inductance circuit as previously discussed need to be modified to correct for the presence of the lead inductance. The following equations are given without proof, and show how L and M need to be modified to compensate for the grid (or plate) lead inductance.

$$L = \left(\frac{m^2+1}{4m} - \gamma\right)L_k \tag{29}$$

$$M = \left(\frac{m^2-1}{4m} + \gamma\right)L_k \tag{30}$$

where

$$\gamma = \frac{\text{lead inductance}}{L_k}.$$

The effect of the cathode lead inductance is much more serious. This inductance, in conjunction with the grid-to-cathode capacitance, produces an input grid conductance which is equal to[11]

$$G = G_m\omega^2 L_c C_g. \tag{31}$$

The effect of this conductance is discussed in the following section.

(c) The Effect of Grid Losses

At high frequencies, there are two sources of grid loading. One of these was mentioned above, and is due to currents flowing through the grid-to-cathode capacitance and cathode lead inductance. The second of these is the transit-time effect, which also produces resistive loading of the grid circuit. Both of these loading conductances are approximately proportional to frequency squared. The relative importance of the two effects depends upon the tube geometry.

Thus, at the high frequencies, the input resistance approaches the characteristic impedance of the grid line, and the attenuation will rise rapidly. This is shown in (32) which gives the fractional loss of gain due to a grid loading conductance of G in terms of the G_m of the tube, the gain of the stage A, and the normalized slope of the phase function $d\beta/dx_k$.

$$\frac{E_{\text{out}} \text{ with grid losses}}{E_{\text{out}} \text{ without grid losses}} = 1 - \frac{A}{4}\,\frac{G}{G_m}\,\frac{d\beta}{dx_k} \tag{32}$$

$d\beta/dx_k$ is equal to $2/\sqrt{1-x_k^2}$ for a constant-K section, and is approximately equal to 2 for properly designed sections with negative mutual inductance. The derivation for (32) is given in Appendix IV.

[11] F. E. Terman, "Radio Engineering," McGraw-Hill Book Co., New York, N. Y., 1947; pp. 369–371.

VIII. Noise in Distributed Amplifier

There are four sources of noise of basic and unavoidable nature that need to be considered in any amplifier which extends to high frequencies. These are:

(a) Thermal noise in the input impedance.

(b) Shot-effect noise generated by the electron stream in the electron tubes.

(c) Induced grid noise, which is associated with transit time effects at the high frequencies.

(d) Thermal noise in the equivalent grid-loading impedance which is developed between the cathode and grid of an electron tube as a result of grid-to-cathode capacitance and the cathode lead inductance.

The ideal amplifier would be one in which the only noise in the output terminal was due to the thermal noise in the input impedance of the amplifier. The thermal noise in the input impedance can be used as a comparison standard and all other noises can be measured in terms of it.

The manner in which these various noises appear in the output of the distributed amplifier will be considered below. The analysis will be carried out for a single-stage distributed amplifier, shown in Fig. 1.

(a) *Thermal Noise*

The grid line is terminated with resistances on each end, and both of these act as generators of thermal noise. The noise generated in the input termination will cause a noise voltage to appear at the output terminals in exactly the same way as if it were a signal. The noise due to the grid termination produces a noise wave on the grid line which is amplified by the tubes, the noise signals adding in the plate line in a way which depends upon the phase shift per section. The addition of the noise voltages in the plate line due to the backward-going wave is the same mathematical problem as was considered in the case of reflections in Section IV. Calling the noise power in the output due to the input impedance N_1 and noise power due to grid termination N_2,

N_T = total thermal noise output in a band Δf cycles wide at frequency f

$$= N_1 + N_2$$

$$= N_0 \frac{Z_{01}}{Z_{02}} A_0^2 n^2 \left[1 + \left(\frac{\sin n\phi}{n \sin \phi} \right)^2 \right] \tag{33}$$

where

$N_0 = 4kT\Delta f$ watts

k = Boltzman's constant

T = temperature of the terminations, °K

Δf = bandwidth in cps in which noise is to be measured

f = frequency

A_0 = amplification of each section $= G_m Z_{02}/2$

ϕ = phase shift per section

n = number of sections per stage.

The first term in (33) is the amplified noise arising in the input impedance. The second term is due to the noise originating in the grid termination; it can have a value of unity when $\phi = 0$, π, etc., but is, in general, smaller than unity. The functional dependence of this noise power upon the phase shift per section is identical with the square of the voltage reflections from the grid termination shown plotted for $n = 5$ and $n = 11$ in Fig. 12(b). As can be seen from (33) and Fig. 12(b), the thermal noise due to the *grid termination* is usually small compared with the noise due to the input impedance. Only at dc and at cutoff do the two terms become equal.

(b) *Shot-Effect Noise*

The shot-effect noise is due to the random emission of electrons from the cathode. The effect of this noise can be represented by a resistor in the grid circuit which is assigned a value such that this fictitious resistance generates as much noise as is actually observed in the plate circuit of the tube. If the impedance, looking back from the grid toward the input terminals, can be made much higher than this noise resistance, then the noise due to the shot effect will be small compared with the thermal noise. At low frequencies and in narrow-band amplifiers, the input impedance can be made high and, consequently, the shot-effect noise can be made to be negligible. In wide-band amplifiers, including the distributed amplifier, the input impedance cannot be made high, and as a result, the noise generated by the shot effect cannot be neglected.

However, in the case of the distributed amplifier, the shot-effect noise can be made negligibly small in spite of the fact that the grid-to-ground impedance is not high when compared to the equivalent noise resistance. This can be seen from the following considerations. Each tube develops random noise current in its plate circuit independently of the other tubes used in the distributed amplifier. The noise currents cause voltages to appear on the plate line, and these voltages add in the output terminals in a random manner. The random addition of voltages can be obtained by taking a sum of the noise power produced by the individual tubes; thus, if the tubes are alike, the total noise power will be proportional to the number of tubes. On the other hand, the signal at the output terminals is proportional to the number of tubes, and the signal power is proportional to the number of tubes squared. Hence, the signal to noise ratio will be proportional to n, where n is the number of tubes. Thus, by using a sufficient number of sections, it is possible to make the signal as large as one desires compared to the shot-effect noise.

The effect of shot noise can be computed in the usual manner. The following results are given without proof. The shot-effect noise power N_s in the output of the distributed amplifier is

$$N_s = nN_0 A_0^2 \frac{R_{eq}}{Z_{02}} \tag{34}$$

where A_0 is the amplification per section. Thus, for a given tube and desired bandwidth, N_0, A_0, R_{eq}, and Z_{02} are known constants.

(*c*) *High-Frequency Noise*

The transit-time effects and cathode lead inductance can both be taken into account by representing them as shunt resistances from grid to ground in each tube. Associated with this equivalent resistance there is a noise, which can be evaluated in a standard manner.[12]

The behavior of this noise in the output of the distributed amplifier is very complicated. In the first place, the magnitude of the noise is a rapid function of frequency (the noise power per cycle is approximately proportional to frequency squared). In the second place, each tube generates noise voltages which propagate in both direction from the tube. Thus, noise generated by one tube is amplified by all the other tubes. Moreover, this amplification depends upon the particular position of the tube in the distributed amplifier.

Fig. 15 shows a single section of distributed amplifier, indicating the sources of high-frequency noise. While the discussion of the relative magnitudes of the two sources of noise is not within the scope of this paper, it should be pointed out that both effects are determined by the geometric factors within the tube itself. For the purpose of this discussion, it shall be assumed that an equivalent resistance R_A and an accompanying voltage can be found which accounts for the existing noise. If the noise power that R_A can deliver is N_T, then it can be shown that the total noise power N_A due to high-frequency effects in the output is given by

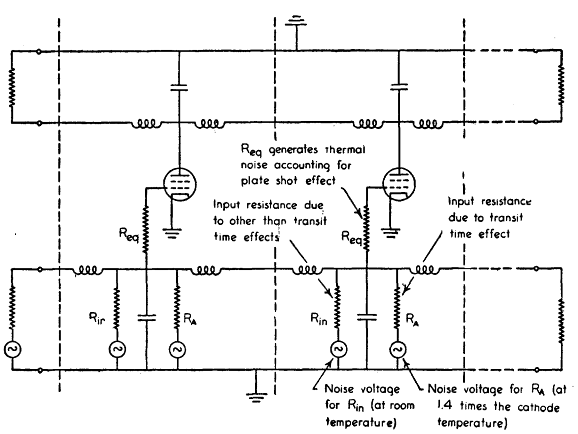


Fig. 15—Sources of noise in section (symbols after Terman).

$$N_A = \frac{N_T R_A A_0^2 Z_{01}^2}{Z_{02}(Z_{01} + 2R_A)^2} \cdot P \qquad (35)$$

[12] See pp. 579–584 of footnote reference 11.

where P is a constant which depends upon ϕ and n. Near dc and near cutoff,

$$\phi \to 0 \quad \text{or} \quad \pi, \quad \text{and} \quad P \to n^3 \cdot \qquad (36)$$

Near midband,

$$\phi \to \frac{\pi}{2}, \quad \text{and} \quad P \to \frac{n^3}{3} \cdot \qquad (37)$$

Thus, it can be seen that the noise power in the output due to grid loading effects is proportional to n^3, whereas the signal voltage is proportional to n^2. Hence, should the noise from this source be at all appreciable, increasing the number of stages decreases the signal-to-noise ratio. However, for reasons having to do with attenuation, this noise is not too important. This will be discussed below.

(*d*) *The Noise Factor of the Distributed Amplifier*

The noise in the output of the amplifier is the sum of the three noises given above:

Total noise power = *thermal noise* + *shot noise* + *grid-loading noise*

or

$$N_{\text{total}} = N_T + N_S + N_A. \qquad (38)$$

The noise factor *N.F.* can be defined as the ratio of the total noise in the output terminals to the noise due to the input impedance. Thus,

$$N.F. = \frac{N_T + N_S + N_A}{N_1} \qquad (39)$$

where the notation is as used above. Substituting values of these terms from (33), (34), and (35), and simplifying,

$$N.F = 1 + \left(\frac{\sin n\phi}{n \sin \phi}\right)^2 + \frac{1}{n} \cdot \frac{R_{eq}}{Z_{01}} + n \frac{Z_{01}}{R_A} \cdot \frac{\alpha}{4} \qquad (40)$$

in which it has been assumed that

(1) $Z_{01} = Z_{02}$, for reasons of simplicity
(2) $R_A \gg Z_{01}$, for reasons to be explained below
(3) α is a numerical factor, equal to about 5, which takes into account the experimentally observed values of noise associated with R_A.

It should also be remembered that R_A is a function of frequency:

$$R_A \propto \frac{1}{f^2} \cdot$$

From (40) it will be seen that noise factor of the amplifier depends upon competing factors of $(1/n)$ and n. Thus, one would think that there should be an optimum value of n for minimum noise. Actually, such a choice would have little physical meaning. In the first place, R_A is a function of frequency; and in the second place, if frequency response is to be at all uniform, one must choose tubes in which $R_A \gg Z_{02}$ at the highest frequency of interest *in order to avoid attenuation*. Under these conditions, the associated high-frequency noise will also be small. Therefore, by using a sufficient number of sections, the shot noise can be made negligibly small and the resulting noise factor can be made to approach unity except at low and high frequencies where it approaches 2 due to the noise arising in the grid termination.

IX. Conclusions

The amplifier described in this paper utilizes the principle of distributing amplification in space, and to this extent bears some relation to the traveling-wave tube. However, it is basically different in its principle of operation and in its field of application. It will permit the construction of wide-band amplifiers with top cutoff frequencies far in excess of those previously obtainable by conventional means. New tubes will undoubtedly be developed specifically for this application and should be characterized by good physical separation between grid and plate terminals, preferably with a ground plane in-between, to which the screen, cathode, and heater may be by-passed. The gain versus bandwidth index of such a tube should be as high as possible, and tubes should present a minimum of grid loading. Present tubes do, in part, met these requirements, but it is felt that if tubes are specifically designed for this purpose, improved performance can be obtained. The techniques herein outlined, although presented in specific detail, are capable of much broader applications. It does not appear necessary to confine the principle of the distributed amplifier to tetrodes alone but should be applicable to other types of amplifier tubes, such as velocity-modulation devices.

Experiments have been conducted which have verified the predictions given in this paper. For example, a two-stage amplifier, using seven 6AK5 tubes per stage with a frequency response of essentially 0 to 200 Mc, had a gain of 18 db. Several such amplifiers will be described in a forthcoming paper which will present experimental confirmation of the principles presented here.

Appendix I

Gain Relations

Fig. 1 shows the basic circuit of a distributed amplifier of the low-pass type. The purpose of this Appendix is to prove the gain relations stated in Sections III and IV.

It will be assumed that it is possible to match impedances between the generator and the grid transmission line and between stages.

If the voltage that is applied to the grid line is e, then the current that will flow in each plate circuit will be eG_m. The impedance that appears between the plate and cathode of each tube is $Z_{02}/2$. Thus, the voltage developed by a single tube is $eG_mZ_{02}/2$. Hence, the gain of the stage is

$$A = \frac{nG_mZ_{02}}{2}. \tag{41}$$

However, if such stages are to be cascaded, then, in general, a transformer must be provided to match the plate line to the grid line of the next stage. Thus, the voltage at the grid of the next stage will be $neG_mZ_{02}/2 \cdot \sqrt{Z_{01}/Z_{02}}$. Hence, the gain of a single stage *measured from grid line to grid line* is

$$A = \frac{nG_m}{2}\sqrt{Z_{02}Z_{01}}. \tag{42}$$

If such stages are cascaded m times, then the resultant gain of the cascaded stages will be

$$G = A^m = \left(\frac{nC_m}{2}\sqrt{Z_{01}Z_{02}}\right)^m. \tag{43}$$

This is (4), given in Section IV.

One can now make use of the fact that Z_{01} and Z_{02} are not really independent variables. More fundamental parameters are: grid-to-cathode capacitance C_g, plate-to-cathode capacitance C_p, and the desired cutoff frequency f_c. Using these, the characteristic impedance of the transmission lines can be written in terms of f_c, C_p, and C_g. It then follows that

$$A = \frac{n}{2f_c}\cdot\frac{G_m}{\pi\sqrt{C_gC_p}}. \tag{44}$$

Wheeler's bandwidth index frequency f_0 was defined in Section III. Using this definition, (43) and (44) become

$$A = n\frac{f_0}{2f_c} \tag{45}$$

$$G = \left(n\cdot\frac{f_0}{2f_c}\right)^m. \tag{46}$$

The total number of electron tubes in a cascaded amplifier is $N = mn$. It is desired to determine the least number of tubes required to produce a given gain. This can be done as follows:

If the gain per stage is

$$G^{1/m} = n\cdot\frac{f_0}{2f_c}, \tag{47}$$

solving for n,

$$n = \frac{2f_c}{f_0}G^{1/m}.$$

Hence,

$$= mG^{1/m}.\frac{2f_c}{f_0}. \tag{48}$$

Differentiating (48) with respect to m and setting the resultant to zero, one finds that the smallest N is obtained when

$$m = \log_\epsilon G. \tag{49}$$

From this and (47), it follows that the corresponding number of sections per stage n is

$$n = \frac{2f_c}{f_0}\epsilon. \tag{50}$$

This is (3) given in Section III. From (49), it follows that, for optimum utilization of tubes, the gain of each stage should be ϵ.

Appendix II

Negative Mutual-Inductance Connection

If m-derived coupling sections are to be used as shown in Fig. 16, it is necessary to calculate the transfer characteristic; i.e., voltage developed per section of plate line per volt in grid line.

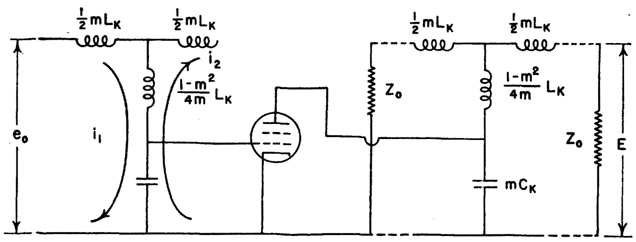


Fig. 16—Negative mutual-inductance connection and symbols.

The grid-drive voltage e_g is given by,

$$e_g = (i_1 - i_2)\cdot\frac{1}{j\omega C_g} = \frac{e_0}{Z_0}(1 - \epsilon^{-j\theta})\cdot\frac{1}{j\omega C_k m} \tag{51}$$

where

$$Z_0 = R_0\sqrt{1 - x_m^2}, \qquad x_m = \pi f R_0 C_k \tag{52}$$

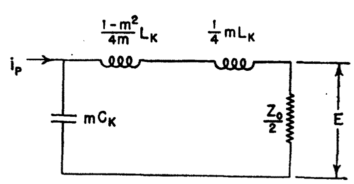


Fig. 17—Equivalent plate circuit of the negative mutual-inductance connection.

and

$$\theta = 2\tan^{-1}\frac{mx_m}{\sqrt{1 - x_m^2}},$$

the phase shift per section of line, but

$$(1 - \epsilon^{-j\theta}) = \frac{2jmx_m}{\sqrt{1 - x_m^2} + jmx_m},$$

or

$$\frac{e_g}{e_0} = \frac{1}{\sqrt{1-x_m^2}}\,\frac{1}{\sqrt{1-(1-m^2)x_m^2}}\angle - \tan^{-1}\frac{mx_m}{\sqrt{1-x_m^2}}\cdot \tag{53}$$

The voltage developed per section of plate line may be readily calculated from the redrawn plate circuit shown in Fig. 17.

$$E = i_p\cdot\frac{\dfrac{Z_0}{2}\cdot\dfrac{1}{j\omega mC_k}}{\dfrac{1}{j\omega mC_k} + \dfrac{Z_0}{2} + j\omega\dfrac{L_k}{4m}}$$

mx_m

$$= \frac{i_pR_0}{2}\,\frac{}{\sqrt{1 - (1 - m^2)x_m^2}}\angle - \tan^{-1}\frac{mx_m}{\sqrt{1 - x_m^2}} \tag{54}$$

but

$$i_p = e_gG_m.$$

Thus the transfer characteristic is given by

$$\frac{E}{e_0} = \frac{G_mR_0}{2}\cdot\frac{1}{\sqrt{1 - x_m^2}[1 - (1 - m^2)x_m^2]}\angle - \theta. \tag{55}$$

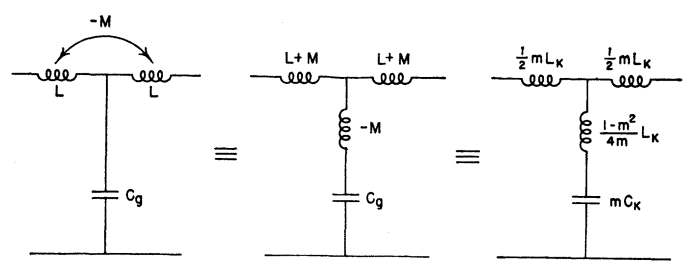
The delay per section τ is given by

$$\tau = \frac{d\theta}{d\omega} = \frac{d\theta}{dx_m}\cdot\frac{1}{\omega_c}$$

$$= \frac{2}{\omega_c}\,\frac{m}{\sqrt{1 - x_m^2}[1 - (1 - m^2)x_m^2]}\cdot \tag{56}$$

Equating the physical structure against the desired structure as shown in Fig. 18, it is evident that

$$M = \frac{m^2 - 1}{4m}L_k, \quad L = \frac{m^2 + 1}{4m}L_k, \quad C_g = mC_k. \tag{57}$$



(a) (b) (c)

Fig. 18—Negative mutual inductance connection and its m-derived equivalents.

In a normal constant-k section not m-derived,

$$x_k = \pi fR_0C_k = \pi fR_0C_g. \tag{58}$$

In the above m-derived structure,

$$x_m = \pi fR_0C_k = \pi fR_0\frac{C_g}{m} = \frac{x_k}{m}\cdot \tag{59}$$

Then substituting $x_m = x_k/m$ in the equations for amplitude response, phase shift, and phase delay so that the results may be compared to constant-k operation, it is found that,

$$\frac{E}{e_0} = \frac{G_m R_0}{2} \frac{m^3}{[m^2 - (1 - m^2)x_k^2]\sqrt{m^2 - x_k^2}} \angle -\theta_2 \quad (60)$$

$$\theta = 2 \tan^{-1} \frac{m x_k}{\sqrt{m^2 - x_k^2}} \quad (61)$$

$$\tau = \frac{1}{\pi f_c} \frac{m^3}{[m^2 - (1 - m^2)x_k^2]\sqrt{m^2 - x_k^2}} \quad (62)$$

where f_c in (62) is equal to

$$f_c = \frac{1}{\pi R_0 C_g} \cdot$$

Appendix III

The Bridged-Tee Connection

The bridged-tee structure shown in Fig. 19(a) may, by Bartlett's bisection theorem,[13] be equated to the lattice section shown in Fig. 19(c). The characteristic impedance Z_0 is, however, given by

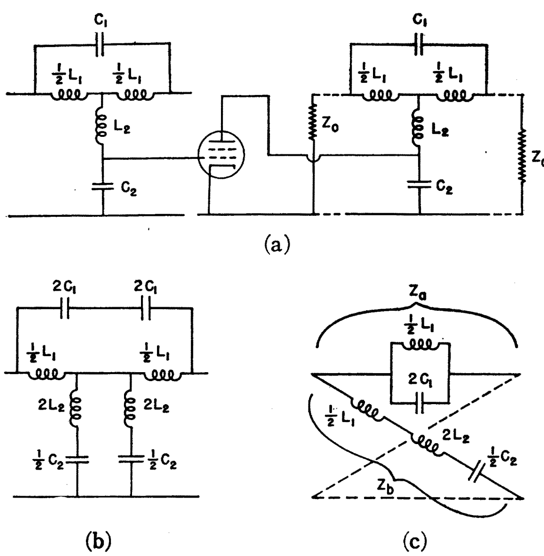


Fig. 19—Bridged-tee connection and symbols.

$$Z_0 = \sqrt{Z_a Z_b} = \sqrt{\frac{L_1}{C_2} \frac{1 - \omega^2 \frac{L_1 C_2}{4}(1 + \alpha)}{1 - \omega^2 L_1 C_1}} \quad (63)$$

where

$$\alpha = 4 \frac{L_2}{L_1} \cdot$$

If $C_2(1+\alpha) = 4C_1$, Z_0 is independent of frequency and equal to $\sqrt{L_1/C_2}$.

The propagation function γ of a lattice is defined as

$$\gamma = \log_\epsilon \left[\frac{1 + \sqrt{\frac{Z_a}{Z_b}}}{1 - \sqrt{\frac{Z_a}{Z_b}}} \right] = 2 \tanh^{-1} \sqrt{\frac{Z_a}{Z_b}}, \quad (64)$$

but when

$$C_2(1 + \alpha) = 4C_1, \qquad \sqrt{\frac{Z_a}{Z_b}} = \frac{j x_k}{1 - x_k^2(1 + \alpha)}$$

where x_k is defined as before as $\pi f R_0 C_g$, recognizing that $C_2 \equiv C_g$.

As $\sqrt{Z_a/Z_b}$ is always imaginary, the propagation function γ is imaginary and thus represents only a phase shift with no attenuation, i.e., an all-pass section. The phase shift θ is then

$$\theta = 2 \tan^{-1} \frac{x_k}{1 - x_k^2(1 + \alpha)}, \quad (65)$$

and the delay τ is

$$\tau = \frac{d\theta}{d\omega} = \frac{1}{\pi f_c} \frac{1 + x^2(1 + \alpha)}{[1 - x_k^2(1 + \alpha)]^2 + x_k^2} \text{ seconds.} \quad (66)$$

The grid drive is calculated in the same fashion as in Appendix II with the exception that part of the input current flows in the bridging arm as shown in Fig. 20.

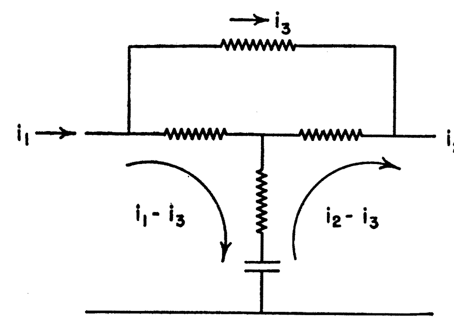


Fig. 20—Bridged-tee current connection.

Thus, the net current through the capacitors is

$$(i_1 - i_3) - (i_2 - i_3) = (i_1 - i_2)$$

And so

$$e_g = (i_1 - i_2) \cdot \frac{1}{j\omega C_2} \quad (67)$$

or

$$\frac{e_g}{e_0} = \frac{1}{j\omega C_2 Z_0} (1 - \epsilon^{-j\theta}) \quad (68)$$

or

$$\frac{e_g}{e_0} = \frac{1}{\sqrt{[1 - x_k^2(1+\alpha)]^2 + x_k^2}} \angle -\tan^{-1} \frac{x_k}{1 - x_k^2(1+\alpha)} \cdot \quad (69)$$

The voltage developed per section of plate line may

[13] Bartlett's bisection theorem states that any symmetrical network may be represented by an equivalent lattice network in which one arm of the lattice is equal to the bisected symmetrical section open-circuited on the bisected end and the other arm of the lattice is equal to the bisected symmetrical section short-circuited.

be readily calculated from the redrawn plate circuit shown in Fig. 21. As no voltage difference exists across

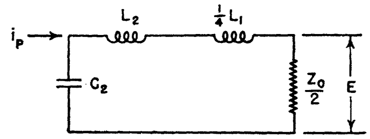


Fig. 21—Equivalent plate circuit of the bridged-tee connection.

the two ends of the bridging arm due to the current i_p, it may be omitted, allowing the series arms and terminating resistors to be combined in parallel. Thus,

$$E=i_p\cdot\frac{\dfrac{Z_0}{2}\cdot\dfrac{1}{j\omega C_2}}{\dfrac{Z_0}{2}+\dfrac{1}{j\omega C_2}+\dfrac{1}{4}j\omega L_1(1+\alpha)}$$

$$=\frac{i_pR_0}{2}\,\frac{1}{\sqrt{[1-x_k^2(1+\alpha)]^2+x_k^2}}\angle-\tan^{-1}\frac{x_k}{1-x_k^2(1+\alpha)}$$

but

$$i_p = e_gG_m. \tag{70}$$

Thus, the transfer characteristic is given by

$$\frac{E}{e_0}=\frac{G_mR_0}{2}\,\frac{1}{[1-x_k^2(1+\alpha)]^2+x_k^2}<-\theta. \tag{71}$$

Equating the physical structure against the desired structure as shown in Fig. 22, it is evident that $L+M=\frac{1}{2}L_1$ and $-M=L_2$. Therefore,

$$L = L_1/2 + L_2 = L_1/2\left(1+\frac{\alpha}{2}\right) \quad \text{as} \quad L_2 = \alpha L_1/4,$$

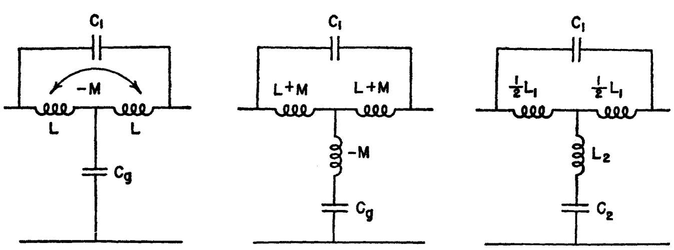


Fig. 22—Bridged-tee connection and its equivalents.

the coefficient of coupling k is

$$k=\frac{M}{L}=-\frac{\alpha}{2+\alpha}, \tag{72}$$

or

$$\alpha+1=\left(\frac{1-k}{1+k}\right). \tag{73}$$

$$C_1=C_2\frac{1+\alpha}{4}=\frac{1+\alpha}{4}C_g=\frac{1}{4}\left(\frac{1-k}{1+k}\right)C_g. \tag{74}$$

Thus, the transfer characteristic may be given as,

$$\frac{E}{e_0}=\frac{G_mR_0}{2}\,\frac{1}{\left[1-x_k^2\left(\dfrac{1-k}{1+k}\right)\right]^2+x_k^2}\angle-\theta \tag{75}$$

$$\theta=2\tan^{-1}\frac{x_k}{1-x_k^2\left(\dfrac{1-k}{1+k}\right)}, \tag{76}$$

and the delay τ as,

$$\tau=\frac{1}{\pi f_c}\,\frac{1+x_k^2\left(\dfrac{1-k}{1+k}\right)}{\left[1-x_k^2\left(\dfrac{1-k}{1+k}\right)\right]^2+x_k^2}\ \text{seconds.} \tag{77}$$

From these equations, curves may be plotted as a function of x_k for various values of the design parameter k, the coefficient of coupling.

Appendix IV

Attenuation Due to Grid Losses

The effect of a shunting conductance G across the shunt capacitor C will introduce an attenuation per section given by

$$\alpha\approx\frac{G}{4\pi f_cC}\,\frac{d\theta}{dx_k} \tag{78}$$

where

θ = the phase shift per section
f_c = the cutoff frequency
x_k = the normalized frequency function.

If the voltage e_0 is applied to the first section of the grid line, the output voltage E of an n-section stage will be given by

$$E=e_0\frac{G_mR_0}{2}\left[1+\epsilon^{-\alpha}+\epsilon^{-2\alpha}\cdots\epsilon^{-(n-1)\alpha}\right]$$

$$=e_0\frac{G_mR_0}{2}\,\frac{1-\epsilon^{-n\alpha}}{1-\epsilon^{-\alpha}}$$

$$\approx e_0\frac{G_mR_0n}{2}\left(1-\frac{n\alpha}{2}\right) \tag{79}$$

R_0 is, however, equal to $1/\pi f_cC$, and thus

$$n=\frac{2A}{G_mR_0}=\frac{2\pi f_cCA}{G_m} \tag{80}$$

where A is the stage gain, neglecting losses.

Thus the fractional loss in gain $n\alpha/2$ is given by

$$\frac{n\alpha}{2}=\frac{A}{4}\,\frac{G}{G_m}\,\frac{d\theta}{dx_k}. \tag{81}$$