Single Capacitor Band-pass Filter.

1 Transfer function Derivation.

The schematic of a configuration that purports to be a bandpass filter is shown in figure 1(a).

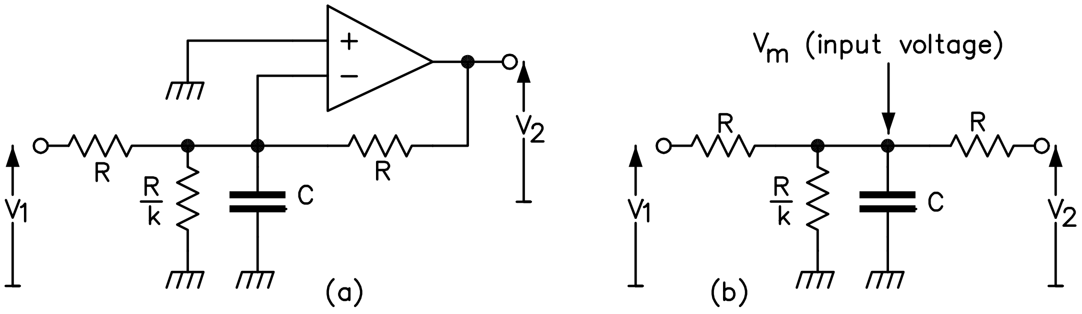


Figure 1: (a) Bandpass filter schematic. (b) Removing the op-amp before applying Millman's theorem.

We will see that k, a parameter scaling the resistor in parallel with the capacitor, determines filter Q. It can be omitted, but then both peak gain and Q rise to several hundred, too high to be useful.

The conventional method for analysing an op-amp filter circuit is to

- Assume an "ideal" operational amplifier having infinite gain,
- Find the relationship between the input and output voltages which makes the voltage at the inverting input zero.

But the "ideal" op-amp assumption normally invoked won't work here. This would imply that the op-amp input voltage, $V_m = 0$, since an ideal op-amp is assumed to have infinite gain. Applying Millman's theorem to find V_m then gives

$$\frac{\dfrac{V_1}{R} + \dfrac{V_2}{R}}{\dfrac{2}{R} + \dfrac{k}{R} + sC} = V_m = 0 \tag{1}$$

$$\text{multiplying top and bottom by } R, \quad \frac{V_1 + V_2}{2 + k + sCR} = 0 \tag{2}$$

$$\text{The denominator value is immaterial, so} \quad V_2 = -V_1 \tag{3}$$

Thus including the capacitor apparently has no effect, and we have a simple inverting circuit. However, if the op-amp gain is assumed to be finite, with a typical single dominant-pole frequency roll-off characteristic, the right-hand side is no longer zero, and we find a different expression.

We can approximate the frequency-dependent voltage gain of the op-amp by the expression

$$A_v(\omega) = -\frac{A_o}{1 + j\dfrac{\omega}{\omega_0}} \tag{4}$$

$$\text{where } A_v(\omega) = \text{the open-loop gain as a function of } \omega, \tag{5}$$

$$A_o = \text{the voltage gain at } dc, \tag{6}$$

$$\omega_0 = \text{the 3 dB angular roll-off (3 dB) frequency of the op-amp.} \tag{7}$$

This 3 dB frequency is of the order of 10 Hz (or $\omega_o \approx 60$), much lower than the resonant response frequency formed, which is typically around 600 - 1500 Hz ($\omega_o \approx 3800 - 9000$ radian/second).

$$\text{If } \omega >> \omega_0,\ A_v(\omega) \approx -\frac{A_o}{j\omega/\omega_0} = -\frac{A_o\omega_o}{j\omega} = -\frac{A_o\omega_o}{s} \tag{8}$$

Relating V_m and V_2 by the voltage gain, we find

$$\begin{aligned}
V_2 &= -V_m\frac{A_o\omega_o}{s} = -V_m\frac{B}{s} && (9)\\
\text{or } V_m &= -V_2\frac{s}{B} && (10)\\
\text{where } B &= A_o\omega_o, && (11)\\
&= \text{ the angular frequency gain-bandwidth product of the op-amp.} && (12)
\end{aligned}$$

Now, applying Millman's theorem again to find the input voltage V_m as a function of V_1 and V_2,

$$\begin{aligned}
\frac{\frac{V_1}{R}+\frac{V_2}{R}}{\frac{2}{R}+\frac{k}{R}+sC} &= V_m = -V_2\frac{s}{B} && (13)\\
\text{simplifying, } \frac{V_1+V_2}{2+k+sCR} &= -V_2\frac{s}{B} && (14)\\
V_1+V_2 &= -V_2\frac{s}{B}(2+k+sCR) && (15)\\
V_1 &= -V_2\left[\frac{s(2+k)}{B}+\frac{s^2CR}{B}+1\right] && (16)\\
&= -V_2\left[s^2+\frac{s(2+k)}{RC}+\frac{B}{RC}\right]\frac{RC}{B} && (17)\\
\text{whence } \frac{V_2}{V_1} &= -\frac{B/RC}{s^2+\frac{s(2+k)}{RC}+\frac{B}{RC}} && (18)\\
\text{which is of the form } \frac{V_2}{V_1} &= -\frac{\omega_o^2}{s^2+\frac{\omega_o}{Q}+\omega_o^2} && (19)\\
\text{where } \omega_0 &= \left(\frac{B}{RC}\right)^{1/2} && (20)\\
\text{and } \frac{\omega_o}{Q} &= \frac{2+k}{RC} && (21)\\
\text{or } f_o &= \frac{1}{2\pi}\left(\frac{B}{RC}\right)^{1/2} \text{ Hz} && (22)\\
\text{and eliminating } \omega_0,\ Q &= \frac{1}{2+k}(RCB)^{1/2} && (23)
\end{aligned}$$

Equations 18, 22 and 23 give expressions for the transfer function, the resonant frequency, f_o and Q.

Re-writing equation 18 in the standard form of equation 19 shows it to be a *low-pass second-order* function. This is counter-intuitive, because casual inspection of the schematic of figure 1(a) shows only one capacitor, and thus we would simplistically expect it to achieve only ±20 dB/decade Bode-plot slopes, whereas this second-order function implies a 40 dB/decade high-frequency asymptotic roll-off.

Both f_0 and Q are a function of not only the RC product, but also the gain-bandwidth product, B, and are thus op-amp dependent! There is a slight wrinkle here. A low-pass high-Q function does *not* have its maximum response at f_0, but slightly lower, at f_m, where[1]

$$f_m = f_o\left(1 - \frac{1}{2Q^2}\right)^{1/2} \tag{24}$$

2 Validation of the Results.

Assume a "typical" op-amp having $A_o = 10^5$ and $f_{3dB} = 10\,\mathrm{Hz}$. Then

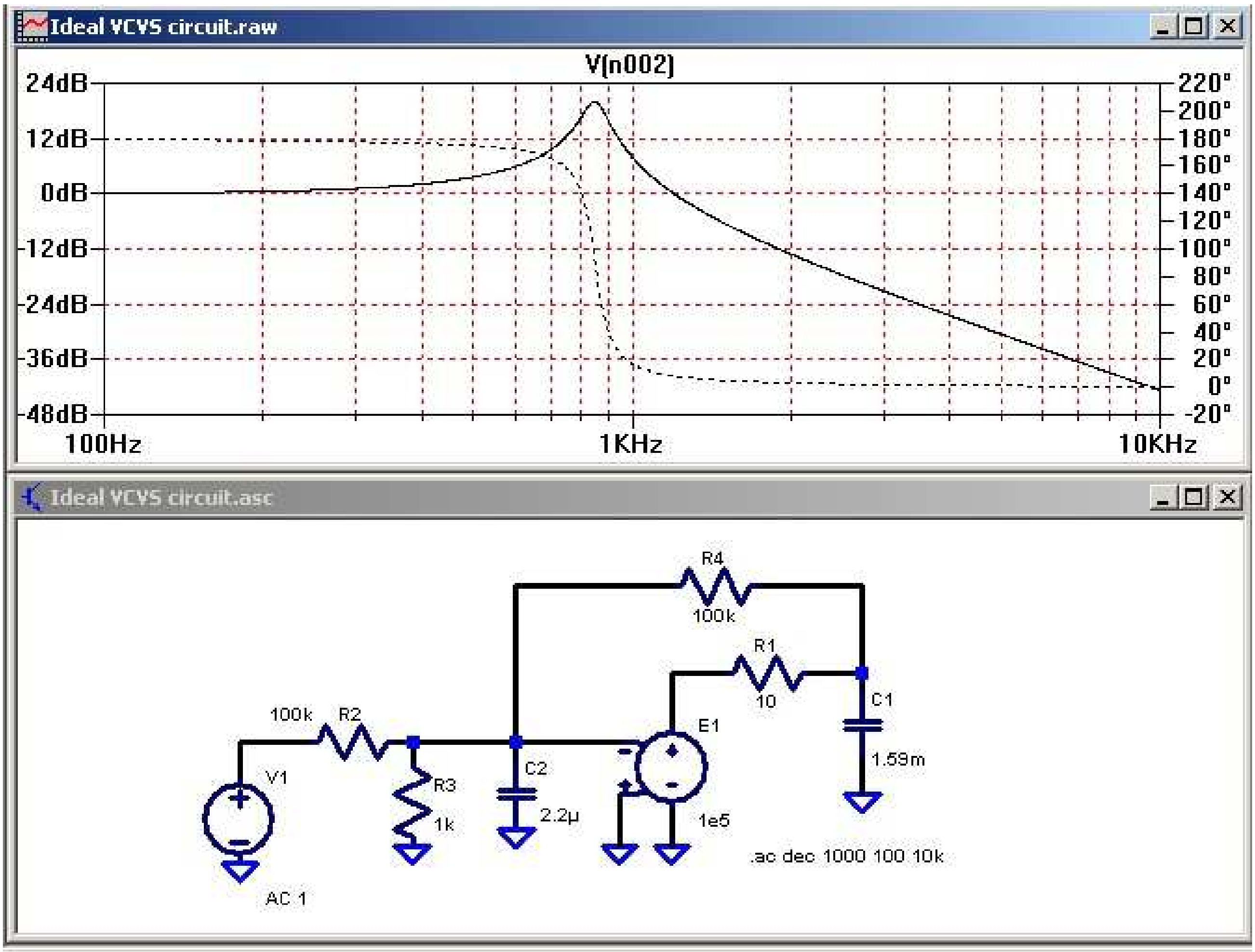
$$B = 2\pi \times 10^5 \times 10 = 6.28 \times 10^6. \tag{25}$$

$$\text{assume } R = 100\mathrm{k}\Omega, \quad C = 2.2\,\mu\mathrm{F}. \quad k = 100 \tag{26}$$

$$\text{from equation 22, } f_o = 850.5\,\mathrm{Hz} \tag{27}$$

$$\text{from equation 23, } Q = 11.5 \tag{28}$$

The figure below shows, bottom window, an LTspice simulation. The op-amp is simulated by an ideal voltage-controlled voltage source (VCVS) having $A_o = 10^5$ and an output RC filter designed for a 3 dB frequency of 10 Hz. The low R value (10 Ω) is chosen to give an output impedance always much less than the impedances it's required to drive, as implemented by a "real" op-amp.



The output voltage plot shows that the theoretical expressions for f_o and Q are correct. The response rolls off asymptotically above the maximum at 40 dB/decade, as expected from the transfer function.

[1] Prove this by differentiating the magnitude of the output voltage with respect to ω, equating the result to zero, and solving for ω_m. The frequency difference is less than 3% for $Q > 10$.

Figure 2: Circuit simulation using an ideal VCVS.

We now substitute a real op-amp, a *Linear Technology* LT1001A, in the circuit of the lower window of figure 4, using the same component values. See the figure below.

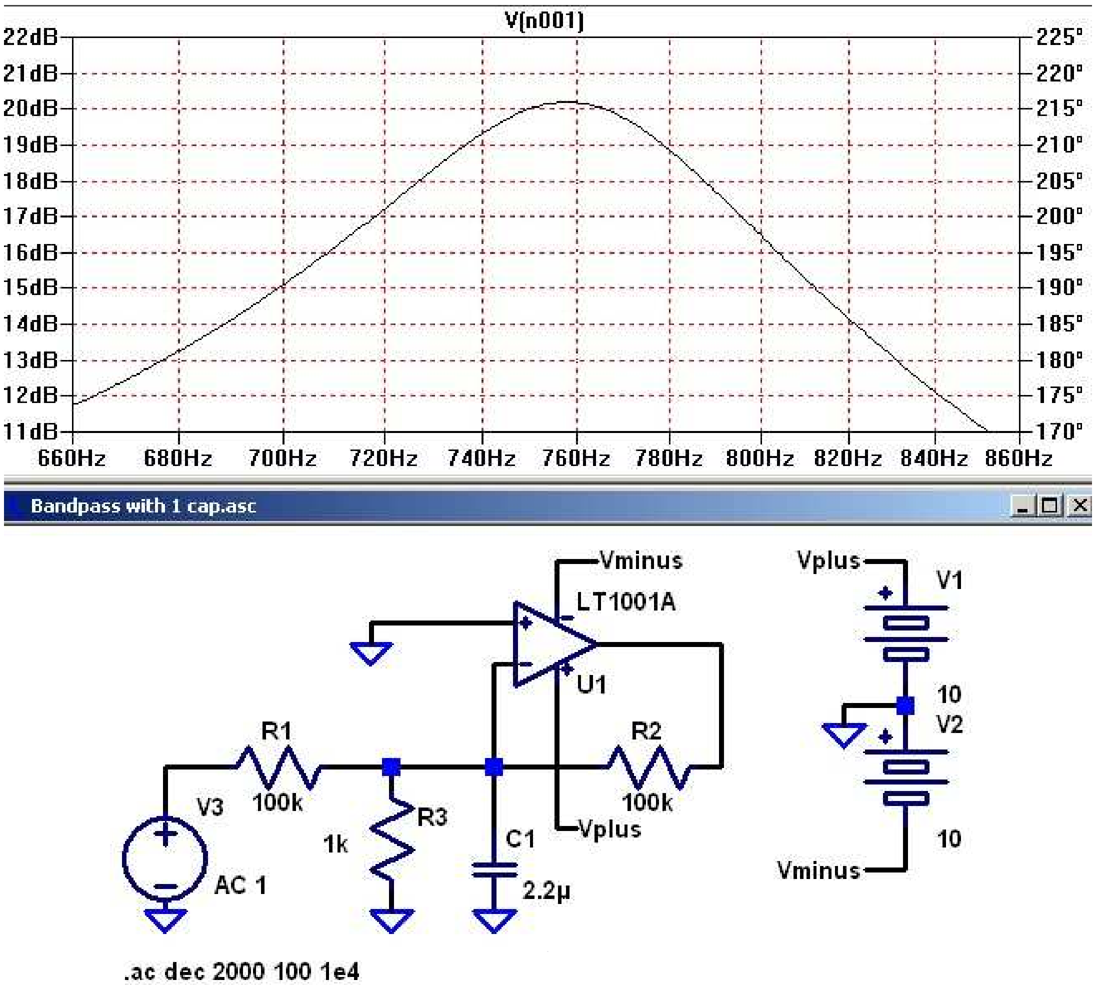


Figure 3: Circuit simulation using a real LT1001A op-amp.

Here, the transfer function plot in the upper window has been zoomed to inspect the response peak. The frequency of maximum response has dropped to about 758 Hz. This corresponds to an angular frequency gain-bandwidth product of about 5×10^7, which from the data-sheet seems reasonable. Other op-amps give different responses, those with higher gain-bandwidth products producing higher f_o and Q as expected from the theoretical expressions.

An Octave Simulation.

The theoretical voltage gain given by equation 19 can be plotted to verify its form. Below is an `Octave` code to do this, for normalized $\omega_o = 1$. The label font size has been increased for greater legibility, and a finer grid used. The plot can be zoomed for inspection. The Postscript file used for this document was produced with the `-dps` command, then "eps clipped" in *Ghostview*.[2]

[2] All software used for modelling, simulation, and document preparation (*LaTeX*) is freeware.

```
% "lpq" plots lo-pass filter characteristic
% GEJB 6 September 2010

Q = input("Q value?  ");
w = logspace(-1, 1, 20000);  s = j*w;  % Make 20k point 's' vector
Av = 1./(s.^2 + s/Q + 1);              % Find complex voltage gain
set (0, "defaultaxesfontsize", 18);    % Increase font size for visibility
semilogx(w, abs(Av)); grid("minor");   % Put in finer grid
xlabel('Log Frequency.'}; ylabel('Response, dB');
```

Below is shown the resulting plot for $Q = 10$.

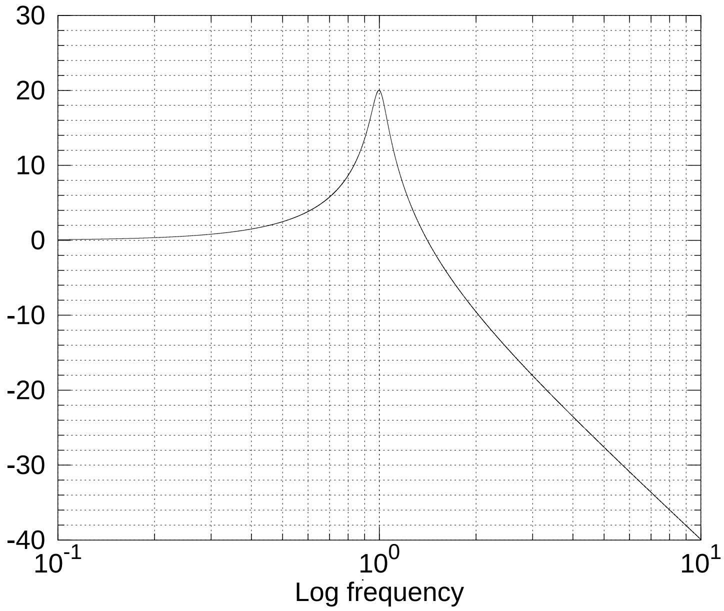


Figure 4: Circuit simulated using *Octave*.

3 Conclusions.

- The additional phase shift of $\pi/2$ radian, required for a second order transfer function, is supplied by the op-amp, which has an asymptotic phase shift of this value as its gain rolls off at 20 dB/decade.
- The shape of the transfer function depends not only on the RC product, but also on the op-amp gain-bandwidth product, and hence is device-dependent. The selectivity of the resonant portion of the characteristic can be increased by decreasing k.
- In theory, such a circuit having a maximum phase shift around the feedback loop of π radian is unconditionally stable[3], but a "real" opamp is more complicated than the simple model used here, and stray circuit L and C may cause instability.
- In principle, you could use this simple circuit, followed by an amplifier/buffer, for a CW filter. I haven't tried this yet, but I will.

[3]See Bold and Tan, "*Theoretical and computer analysis of circuits and systems*" page 127.